GNS-pair: construction and applications

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Outline

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- Fix $n \geq 1$ and let $\{x_1, x_2, \cdots, x_n\} \subseteq \mathcal{X}$. Then

$$\begin{aligned}\sum_{i,j=1}^{n} \overline{\lambda_i}\lambda_j \mathtt{k}(x_i, x_j) &= \sum_{i,j=1}^{n} \overline{\lambda_i}\lambda_j \langle \wp(x_i), \wp(x_j) \rangle \\ &= \Big\| \sum_{i=1}^{n} \lambda_i \wp(x_i) \Big\|^2 \\ &\geq 0\end{aligned}$$

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  for all $\{\lambda_1, \lambda_2, \cdots, \lambda_n\} \subseteq \mathbb{C}$.
- Thus there exists a "positive definite kernel" $\mathtt{k}$ on $\mathcal{X}$.

Positive definite kernel

Definition: A positive definite kernel (p.d.k) on a set $\mathcal{X}$ is a map $\mathtt{k} : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ satisfying

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Example:

- Suppose $\mathcal{H}$ is a Hilbert space and let $\mathcal{X} = \mathcal{H}$.
- Define $\mathtt{k} : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ by $\mathtt{k}(x, y) = \langle x, y \rangle$.
- $\sum_{i,j=1}^{n} \overline{\lambda_i} \lambda_j \mathtt{k}(x_i, x_j) = \sum_{i,j=1}^{n} \overline{\lambda_i} \lambda_j \langle x_i, x_j \rangle$
  $= \langle \sum_i \lambda_i x_i, \sum_j \lambda_j x_j \rangle = \left\| \sum_i \lambda_i x_i \right\|^2 \geq 0.$
- $\mathtt{k}$ is a p.d.k.

GNS-pair $(\mathcal{H}, \wp)$

Theorem:(GNS-construction) Let $\mathcal{X}$ be a set and $\mathtt{k}$ be a p.d.k on $\mathcal{X}$. Then there a Hilbert space $\mathcal{H}$ and a map $\wp : \mathcal{X} \to \mathcal{H}$ such that

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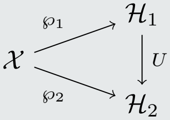
- By construction $\mathcal{H} = \overline{\text{span}}\{\wp(x) : x \in \mathcal{X}\}$.

Uniqueness of GNS-pair

<u>**Theorem**</u>: If $(\mathcal{H}_i, \wp_i),\ i = 1, 2$, are two GNS-pair associated to a p.d.k $\mathtt{k}$ on a set $\mathcal{X}$, then there exists a unitary $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $U \circ \wp_1 = \wp_2$, that is, the following diagram commutes:



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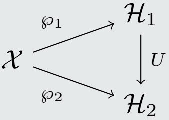

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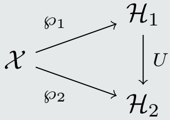

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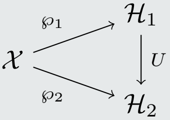

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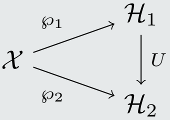

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- $\sum_{i,j=1}^{n} \overline{\lambda_i}\lambda_j \mathtt{k}\big((h_{1i}, h_{2i}), (h_{1j}, h_{2j})\big)$
  $= \langle \sum_i \lambda_i h_{1i}, \sum_j \lambda_j h_{1j} \rangle + \langle \sum_i \lambda_i h_{2i}, \sum_j \lambda_j h_{2j} \rangle \geq 0.$
- $\mathtt{k}$ is a p.d.k, and hence there exits the (unique) GNS-pair $(\mathcal{H}, \wp)$.

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- Equation $(\dagger)$ implies that $V^n = \begin{bmatrix} T^n & * \\ * & * \end{bmatrix}$ for all $n \geq 1$.

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Theorem: Suppose $T \in \mathcal{B}(\mathcal{H})$ is a contraction, i.e., $\|T\| \leq 1$. Then there exists a isometric dilation $(\mathcal{K}, V)$ of T. Moreover, the dilation can be chosen to be *minimal* in the sense that $\mathcal{K} = \overline{\text{span}}\{V^n h : h \in \mathcal{H}, n \geq 0\}$.

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- $\big\langle h, P_{\mathcal{H}}V^n h\big\rangle = \big\langle h, T^n h\big\rangle$ for all $h, h' \in \mathcal{H}$, hence $T^n = P_{\mathcal{H}}V^n|_{\mathcal{H}}$.
- $V^n h = V^n(\wp(0,h)) = \wp(n,h)$, hence $\mathcal{K} = \overline{\text{span}}\{V^n h : h \in \mathcal{H}, n \geq 0\}$.

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Corollary: Every contraction on a Hilbert space has a minimal unitary dilation.

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- There are lots of questions, answers & problems...

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THANK YOU