Nonnegative tensors and their applications

M. Rajesh Kannan

Department of Mathematics,
University of Manitoba,
Winnipeg, Canada.

17-03-2016

Positivity of homogenous polynomials

- Consider an m^{th}-degree homogeneous polynomial in n variables $P(x_1, \ldots, x_n) = \sum_{i_1,\ldots,i_m=1}^{n} a_{i_1 \ldots i_m} x_{i_1} \ldots x_{i_m}$.

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- Applications in control theory, dynamical systems, statistical analysis of multiarray data, etc.
- Unfortunately, checking the nonnegativity of a polynomial is not a easy task (i.e., there is no easy algorithm).
- *We can associate a symmetric tensor with the given polynomial and checking the nonnegativity of the polynomial is equivalent to positive semidefiniteness of the associated tensor .*

Tensor

Definition

An m-order n-dimensional real *tensor (hypermatrix)* is a multidimensional array of n^m elements of the form

$$\mathcal{A} = (A_{i_1 \ldots i_m}),\ A_{i_1 \ldots i_m} \in \mathbb{R},\ 1 \leq i_1, \ldots, i_m \leq n.$$

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- A square matrix of order n is a 2-order n-dimensional tensor.
- A real tensor is said to be a *nonnegative (positive)* tensor if all its entries are nonnegative (positive).
- A tensor $\mathcal{A}$ is said to be *symmetric* if its entries are invariant under any permutation of the indices $\{i_1, \ldots, i_m\}$.

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For an m-order n-dimensional tensor $\mathcal{A}$ and for a vector $x \in \mathbb{C}^n$,

- $\mathcal{A}x^{m-1}$ is the vector in $\mathbb{C}^n$ defined by

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2,\ldots,i_m=1}^{n} A_{ii_2\ldots i_m} x_{i_2} \cdots x_{i_m}.$$

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- For $x \in \mathbb{C}^n$ and a natural number k, the vector $x^{[k]}$ is the *Hadamard power* of x, i.e. $x_i^{[k]} = x_i^k$ for all i.

Eigenvalues of Tensors

Definition (Qi(2005), Lim(2005))

If a pair $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^n \setminus \{0\}$ satisfies the following equation

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

then λ is called an *eigenvalue* of $\mathcal{A}$ and x is called an *eigenvector* corresponding to the eigenvalue λ.

Eigenvalues of Tensors

$$\mathbb{R}^n_+ = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0 \text{ for all } 1 \leq i \leq n\}$$

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Existence

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The *spectral radius* $\rho(\mathcal{A})$ of a tensor $\mathcal{A}$ is defined to be $\max\{|\lambda| : \lambda \text{ is an eigenvalue of } \mathcal{A}\}$.

Principal subtensor

Definition

Let $\mathcal{A} = (a_{i_1 \ldots i_m})$ be an m-order n-dimensional tensor and $\alpha \subset \{1, \ldots, n\}$ with $|\alpha| = r$. A *principal subtensor* $\mathcal{A}[\alpha]$ of the tensor $\mathcal{A}$ with index set α is an m-order r-dimensional subtensor of $\mathcal{A}$ consisting of r^m elements defined as follows:

$$\mathcal{A}[\alpha] = (a_{i_1 \ldots i_m}) \text{ , where } i_1, \ldots, i_m \in \alpha.$$

Theorem (Chang, Pearson and Zhang(2008))

If $\mathcal{A}$ is an m-order n-dimensional nonnegative tensor, then there exist $\lambda \geq 0$ and a nonnegative vector $x \geq 0$ such that

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}.$$

Irreducible tensors

Definition (Lim(2005))

A tensor $\mathcal{A} = (A_{i_1 \dots i_m})$ of order m dimension n is called *reducible* if there exists a nonempty proper subset $\alpha \subset \{1, \dots, n\}$ such that

$$A_{i_1 \dots i_m} = 0 \text{ for all } i_1 \in \alpha \text{ and } i_2, \dots, i_m \notin \alpha.$$

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Example

Consider the 3-order 2-dimensional tensor $\mathcal{A}$ defined as follows:

$$A_{111} = A_{222} = A_{122} = A_{211} = 1$$

and other entries are zero. Then $\mathcal{A}$ is irreducible.

Perron-Frobenius theorem

Theorem (Chang, Pearson and Zhang(2008))

If $\mathcal{A}$ is an m-order n-dimensional irreducible nonnegative tensor , then there exists a pair $(\lambda, x) \in \mathbb{R}_+ \times \mathbb{R}^n_{++}$ such that:

(1) *λ is an eigenvalue.*

(2) *x is an eigenvector of $\mathcal{A}$.*

(3) *If μ is an eigenvalue with nonnegative eigenvector, then $\mu = \lambda$. Moreover, the nonnegative eigenvector is unique up to a multiplicative constant.*

(4) *If μ is an eigenvalue of A, then $|\mu| \leq \lambda$.*

Weakly irreducible nonnegative tensor

With a nonnegative tensor $\mathcal{A} = (A_{i_1 \ldots i_m})$, we associate the nonnegative $n \times n$ matrix $R(\mathcal{A})$:

$$R(\mathcal{A})_{ij} = \sum_{\{i_2,\ldots,i_m\} \ni j} A_{ii_2 \ldots i_m}.$$

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Definition

A nonnegative tensor $\mathcal{A} = (A_{i_1 \dots i_m})$ is said to be *weakly reducible* if $R(\mathcal{A})$ is a reducible matrix. It is *weakly irreducible* if it is not weakly reducible.

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Theorem

Let $\mathcal{A}$ be a nonnegative tensor. If $\mathcal{A}$ is irreducible, then $\mathcal{A}$ is weakly irreducible.

Frobenius normal form

Theorem (Hu, Huang, Qi(2014))

Let $\mathcal{A}$ be an m-order n-dimensional nonnegative tensor. If $\mathcal{A}$ is weakly reducible, then there exists a partition $\{\alpha_1, \ldots, \alpha_k\}$ of $\{1, \ldots, n\}$ such that every tensor in $\{\mathcal{A}[\alpha_j] : j \in \{1, \ldots, k\}\}$ is weakly irreducible and $A_{ri_2 \ldots i_m} = 0$ for all $r \in \alpha_p$, $i_j \in \alpha_q$ for some $j \in \{2, \ldots, m\}$ and $p > q$.

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Proposition (Hu, Huang, Qi(2014))

Let $\mathcal{A}$ be an m-order n-dimensional weakly reducible nonnegative, and let $\{\alpha_1, \ldots, \alpha_k\}$ be a partition of $\{1, \ldots, n\}$ determined by the above Theorem. Then $\rho(\mathcal{A}) = \rho(\mathcal{A}[\alpha_p])$ for some $p \in \{1, \ldots, k\}$.

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Example

Consider the 3-order 2-dimensional tensor $\mathcal{A}$ defined as follows:

$$A_{111} = 1, A_{222} = 1, \ A_{112} = A_{121} = A_{211} = 4$$
$$A_{122} = A_{221} = A_{212} = 0$$

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Since $A_{122} = 0$, $\mathcal{A}$ is reducible. The partition is $\{\{1\}, \{2\}\}$ and the corresponding subtensors are $A_{111} = 1$ and $A_{222} = 1$. But the nonnegative eigenpairs of $\mathcal{A}$ are

$$(1, (0, 1)^T) \text{ and } (7.3496, (0.5576, 0.4425)^T).$$

Perron-Frobenius theorem for WIN tensors

Theorem (Friedland, Gaubert and Han (2013))

Let $\mathcal{A}$ be an m-order n-dimensional weakly irreducible nonnegative tensor. Then $\rho(\mathcal{A})$ is the unique H^{++}-eigenvalue of $\mathcal{A}$. i.e. any eigenvalue corresponding to a positive eigenvector must be equal to $\rho(\mathcal{A})$.

Theorem (M.Rajesh Kannan, N.Shaked-Monderer and A.Berman)

Let $\mathcal{A}$ be an m-order n-dimensional weakly irreducible tensor, and $\mathcal{A}[\alpha]$ a subtensor of $\mathcal{A}$, $|\alpha| = k < n$. Then $\rho(\mathcal{A}[\alpha]) < \rho(\mathcal{A})$.

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Let $\mathcal{A}$ and $\mathcal{B}$ be two tensors such that $0 \leq \mathcal{A} \leq \mathcal{B}$ and $\mathcal{B}$ is a weakly irreducible tensor. Then,

(a) $\rho(\mathcal{A}) \leq \rho(\mathcal{B})$,

(b) *if* $\mathcal{A} \neq \mathcal{B}$, *then* $\rho(\mathcal{A}) < \rho(\mathcal{B})$.

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Any nonnegative eigenvector of a weakly irreducible nonnegative tensor corresponding to the spectral radius must be positive.

Positive definite tensors

For an m-order n-dimensional tensor $\mathcal{A}$ and a vector $x \in \mathbb{C}^n$, the scalar $\mathcal{A}x^m$ is defined to be

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Definition (Qi(2005))

Let $\mathcal{A}$ be an m-order n-dimensional tensor. Then $\mathcal{A}$ is said to be a *positive semidefinite tensor* if for any vector $x \in \mathbb{R}^n$, $\mathcal{A}x^m \geq 0$, and $\mathcal{A}$ is called a *positive definite tensor* if for any nonzero vector $x \in \mathbb{R}^n$, $\mathcal{A}x^m > 0$.

Examples

Definition (Cauchy tensors)

For a vector $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$ with $c_i \neq 0$ for all i, the *Cauchy tensor* $\mathcal{C}$ associated with c is defined as follows:

$$C_{i_1 \ldots i_m} = \frac{1}{c_{i_1} + \cdots + c_{i_m}},$$

for all $i_1, \ldots, i_m \in \{1, \ldots, n\}$. The vector c is called the generating vector.

Theorem

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- *the tensor $\mathcal{C}$ is positive semidefinite if and only if the vector c is positive .*
- *the tensor $\mathcal{C}$ is positive definite if and only if the components of the vector c are positive and distinct.*

Theorem (Qi(2005))

Let $\mathcal{A}$ be an m-order n-dimensional symmetric tensor. Then,

(a) *$\mathcal{A}$ is positive definite if and only if all its H-eigenvalues are positive.*

(b) *$\mathcal{A}$ is positive semidefinite if and only if all its H-eigenvalues are nonnegative.*

$\mathcal{M}$-tensors

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- If $s > \rho(\mathcal{D})$, then $\mathcal{A}$ is called a *strong $\mathcal{M}$-tensor*.
- A $\mathcal{Z}$-tensor $\mathcal{A} = s\mathcal{I} - \mathcal{D}$ with $\mathcal{D} \geq 0$, is called *weakly irreducible*, if $\mathcal{D}$ is weakly irreducible.

Theorem (Qi(2014))

Let $\mathcal{A}$ be an m-order n-dimensional $\mathcal{Z}$-tensor. Then

(a) *$\mathcal{A}$ is an $\mathcal{M}$-tensor if and only if all its H-eigenvalues are nonnegative.*

(b) *$\mathcal{A}$ is a strong $\mathcal{M}$-tensor if and only if all its H-eigenvalues are positive.*

Theorem (M. Rajesh Kannan, N.Shaked-Monderer, A.Berman)

Let $\mathcal{A}$ and $\mathcal{B}$ be m-order n-dimensional $\mathcal{Z}$-tensors such that $\mathcal{A} \leq \mathcal{B}$.

(a) *Suppose $\mathcal{A}$ is an $\mathcal{M}$-tensor (a strong $\mathcal{M}$-tensor). Then $\mathcal{B}$ is also an $\mathcal{M}$-tensor (a strong $\mathcal{M}$-tensor).*

(b) *Suppose $\mathcal{A}$ is a weakly irreducible $\mathcal{M}$-tensor and $\mathcal{A} \neq \mathcal{B}$. Then $\mathcal{B}$ is a strong $\mathcal{M}$-tensor.*

$\mathcal{H}$-tensor

The *comparison tensor* of an m-order n-dimensional tensor $\mathcal{A}$, denoted by $\mathcal{M}(\mathcal{A})$, is defined as follows:

$$\mathcal{M}(\mathcal{A})_{i_1\ldots i_m} = \begin{cases} |a_{i_1\ldots i_m}| & \text{if } i_1 = \cdots = i_m, \\ -|a_{i_1\ldots i_m}| & \text{otherwise.} \end{cases}$$

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A tensor $\mathcal{A}$ is said to be *an $\mathcal{H}$-tensor (a strong $\mathcal{H}$-tensor)* if $\mathcal{M}(\mathcal{A})$ is an $\mathcal{M}$-tensor (a strong $\mathcal{M}$-tensor).

DD-tensor and GDD-tensor

For an m-order n-dimensional tensor $\mathcal{A}$, define

$$r_i(\mathcal{A}) = \sum_{i_2,\ldots,i_m} |a_{ii_2\ldots i_m}| - |a_{i\ldots i}| \text{ for all } i \in \{1,\ldots,n\}.$$

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$$r_i(\mathcal{A}) = \sum_{i_2,\ldots,i_m} |a_{ii_2\ldots i_m}| - |a_{i\ldots i}| \text{ for all } i \in \{1,\ldots,n\}.$$

- An m-order n-dimensional tensor $\mathcal{A}$ is said to be *diagonally dominant* if $|a_{i\ldots i}| \geq r_i(\mathcal{A})$.
- An m-order n-dimensional tensor $\mathcal{A}$ is said to be *generalized diagonally dominant* if there exists $d_1,\ldots,d_n$ positive such that

$$|a_{i\ldots i}|d_i^{m-1} \geq \sum_{(i_2,\ldots,i_m)\neq(i,\ldots,i)} |a_{ii_2\ldots i_m}|d_{i_2}\ldots d_{i_m},$$

for every i .

Theorem (M. Rajesh Kannan, N. Shaked-Monderer, A. Berman)

Let $\mathcal{A}$ be an m-order n-dimensional tensor. If $\mathcal{A}$ is a weakly irreducible diagonally dominant tensor such that $r_i(A) < |a_{i\ldots i}|$ for at least one i, then $\mathcal{A}$ is a strong $\mathcal{H}$-tensor.

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Let $\mathcal{A}$ be a symmetric tensor. Then $\mathcal{A}$ is an $\mathcal{H}$-tensor if and only if $\mathcal{A}$ is a generalized diagonally dominant tensor.

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Theorem (M. Rajesh Kannan, N. Shaked-Monderer, A. Berman)

Let $\mathcal{A}$ be an m-order n-dimensional symmetric tensor such that m is even and $a_{i\ldots i} \geq 0$ for every $i \in \{1, \ldots, n\}$. If $\mathcal{A}$ is an $\mathcal{H}$-tensor, then $\mathcal{A}$ is positive semidefinite.

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Thank You

Nomenclature

- The outer product (or Segre product) of k vectors $u \in \mathbb{C}^{n_1}$, $v \in \mathbb{C}^{n_2}$,..., $z \in \mathbb{C}^{n_k}$ is defined as

$$u \otimes v \otimes \cdots \otimes z = [u_{i_1} v_{i_2} \ldots z_{i_k}]_{i_1, i_2, \ldots, i_k = 1}^{n_1, n_2, \ldots, n_k}.$$

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- The Segre map

$$\phi : \mathbb{C}^{n_1} \times \mathbb{C}^{n_2} \times \cdots \times \mathbb{C}^{n_k} \longrightarrow \mathbb{C}^{n_1 \times n_2 \times \cdots \times n_k},$$

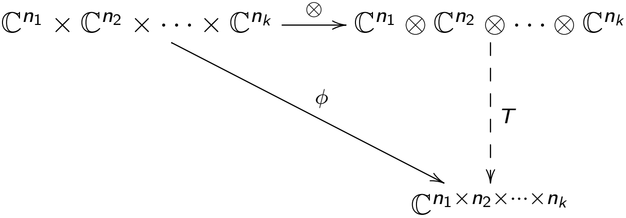
defined as follows

$$\phi(u, v, \ldots, z) = u \otimes v \otimes \cdots \otimes z,$$

is a multilinear map.

Nomenclature

By the universal property of the tensor product, there exists a linear map T such that the following diagram commutes.



Nomenclature

Consider the canonical basis of $\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \cdots \otimes \mathbb{C}^{n_k}$

$$\{e_{j_1}^{(1)} \otimes e_{j_2}^{(2)} \otimes \cdots \otimes e_{j_k}^{(k)} : 1 \leq j_1 \leq n_1, 1 \leq j_2 \leq n_2, \ldots, 1 \leq j_k \leq n_k\},$$

where $\{e_1^{(l)}, \ldots, e_{n_l}^{(l)}\}$ denotes the canonical basis of $\mathbb{C}^{n_l}$, $l = 1, \ldots, k$.

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where $\{e_1^{(l)}, \ldots, e_{n_l}^{(l)}\}$ denotes the canonical basis of $\mathbb{C}^{n_l}$, $l = 1, \ldots, k$.

Then T can be described by

$$T\Big(\sum_{j_1, j_2, \ldots, j_k = 1}^{n_1, n_2, \ldots, n_k} A_{j_1 j_2 \ldots j_k} e_{j_1}^{(1)} \otimes e_{j_2}^{(2)} \otimes \cdots \otimes e_{j_k}^{(k)}\Big) = [A_{j_1 j_2 \ldots j_k}]_{j_1, j_2, \ldots, j_k = 1}^{n_1, n_2, \ldots, n_k}.$$