Compact operators and Hilbert scales in ill-posed problems

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Plan

The lecture consists of two parts:

- Part I
  1. Introduce the concept of Hilbert scales and Gelfand triple;
  2. Construction of Helbert sclaes using Gelfand triples;
- Part II
  1. Ill-posed equations and their regularizations;
  2. Use of Hilbert scales for improving error estimates.

Part I

What are Hilbert scales?

Definition

A family $\{H_s\}_{s\in\mathbb{R}}$ of Hilbert spaces is called a **Hilbert scale** if $H_t \subseteq H_s$ whenever $s < t$ and the inclusion is a continuous embedding, i.e., there exists $c_{s,t} > 0$ such that

$$\|x\|_s \leq c_{s,t}\|x\|_t \quad \forall\, x \in H_t.$$

$\diamondsuit$

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$$\|x\|_s \leq c_{s,t}\|x\|_t \quad \forall\, x \in H_t.$$

$\diamondsuit$

Given a Hilbert space H, a Hilbert scale can constructed by first defining H_s for $s \geq 0$ with $H_0 = H$ and then defining H_s for $s < 0$ using the concept of Gelfand triples.

Gelfand triple

Let H be a Hilbert space and let H_1 be a dense subspace of H, which itself is a Hilbert space with respect to a stronger norm $\|\cdot\|_1$.

For $v \in H$, let

$$\|v\|_{-1} := \sup\{|\langle u, v\rangle| : u \in H_1, \|u\|_1 \leq 1\}.$$

- $v \mapsto \|v\|_{-1}$ is a norm on H and

$$\|v\|_{-1} \leq \|v\| \quad \forall\, u \in H.$$

Let H_{-1} be the completion of H w.r.t. $\|\cdot\|_{-1}$.

- The inclusions $H_1 \subseteq H \subseteq H_{-1}$ are continuous embeddings.

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Definition

The triple (H_1, H, H_{-1}) is called the **Gelfand triple**. $\diamondsuit$

An illustration

Example

Let H be a separable Hilbert space, $\{u_n : n \in \mathbb{N}\}$ be an orthonormal basis of H, and (σ_n) be a sequence of positive real numbers. Let

$$H_1 := \{x \in H : \sum_{n=1}^{\infty} \frac{|\langle x, u_n\rangle|^2}{\sigma_n^2} < \infty\}.$$

Then H_1 is a dense subspace of H, and is a Hilbert space w.r.t. the inner product

$$\langle x, y\rangle_1 = \sum_{n=1}^{\infty} \frac{\langle x, u_n\rangle\langle u_n, y\rangle}{\sigma_n^2}, \quad x, y \in H_1,$$

with stronger norm $x \mapsto \|x\|_1 := \sqrt{\sum_{n=1}^{\infty} \frac{|\langle x, u_n\rangle|^2}{\sigma_n^2}}$. (continues...)

$\diamondsuit$

Example

[Continuation:] For $x \in H$,

$$\|x\|_{-1} := \sup\{|\langle x, u\rangle| : \sum_{n=1}^{\infty} \frac{|\langle u, u_n\rangle|^2}{\sigma_n^2} \le 1\}.$$

Note that for $u \in H_1$ with $\|u\|_1 \le 1$,

$$\begin{aligned} |\langle x, u\rangle| &= \Big|\sum_{n=1}^{\infty} \sigma_n \langle x, u_n\rangle \frac{\langle u_n, u\rangle}{\sigma_n}\Big| \\ &\le \Big(\sum_{n=1}^{\infty} \sigma_n^2 |\langle x, u_n\rangle|^2\Big)^{\frac{1}{2}} \Big(\sum_{n=1}^{\infty} \frac{|\langle u, u_n\rangle|^2}{\sigma_n^2}\Big)^{\frac{1}{2}} \\ &\le \Big(\sum_{n=1}^{\infty} \sigma_n^2 |\langle x, u_n\rangle|^2\Big)^{\frac{1}{2}}. \end{aligned}$$

Thus, $\|x\|_{-1} \le \Big(\sum_{n=1}^{\infty} \sigma_n^2 |\langle x, u_n\rangle|^2\Big)^{\frac{1}{2}}$. ♢

Example

[Example continues:] Let

$$u = \sum_{n=1}^{\infty} \frac{\sigma_n^2 \langle x, u_n \rangle}{\alpha} u_n, \quad \alpha := \Big(\sum_{n=1}^{\infty} \sigma_n^2 |\langle x, u_n \rangle|^2 \Big)^{\frac{1}{2}}.$$

Then $\langle u, u_n \rangle = \frac{\sigma_n^2 \langle x, u_n \rangle}{\alpha}$ so that

$$\sum_{n=1}^{\infty} \frac{|\langle u, u_n \rangle|^2}{\sigma_n^2} = \frac{1}{\alpha^2} \sum_{n=1}^{\infty} \sigma_n^2 |\langle x, u_n \rangle|^2 = 1;$$

$$\langle x, u \rangle = \frac{1}{\alpha} \sum_{n=1}^{\infty} \sigma_n^2 |\langle x, u_n \rangle|^2 = \Big(\sum_{n=1}^{\infty} \sigma_n^2 |\langle x, u_n \rangle|^2 \Big)^{\frac{1}{2}}.$$

Therefore, $\|x\|_{-1} = \Big(\sum_{n=1}^{\infty} \sigma_n^2 |\langle x, u_n \rangle|^2 \Big)^{\frac{1}{2}}.$ ♢

Linear isometry between H_{-1} and $(H_1)'$

We show:

- H_{-1} is linearly isometric with $(H_1)'$. In particukar,
- H_{-1} is a Hilbert spce w.r.t inner product induced from $(H_1)'$.

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Theorem

Let $\widetilde{H} := H$ with $\|\cdot\|_{-1}$. For $v \in \tilde{H}$, let

$$f_v(u) = \langle u, v\rangle, \quad u \in H_1.$$

Then $f_v \in (H_1)'$ and the map $\Phi_0 : \tilde{H} \to (H_1)'$ defined by

$$\Phi_0(v) := f_v, \quad v \in H_{-1},$$

is a linear isometry.

Proof.

Then f_v is linear and

$$|f_v(u)| = |\langle u, v\rangle| \leq \|u\|\,\|v\| \leq \|u\|_1\|v\|, \quad u \in H_1.$$

Hence, $f_v \in (H_1)'$ and

$$\|f_v\| = \sup\{|f_v(u)| : \|u\|_1 \leq 1\} = \|v\|_{-1}.$$

Thus $\Phi_0 : v \mapsto f_v$ is a linear isometry from $\tilde{H}$ to $(H_1)'$. □

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Thus $\Phi_0 : v \mapsto f_v$ is a linear isometry from $\tilde{H}$ to $(H_1)'$. □

We shall extend the map Φ_0 to a surjective linear isometry. First we prove the following.

Theorem

The subspace $\{f_v : v \in H\}$ is dense in $(H_1)'$.

Proof.

By Hahn-Banach extension theorem, it is enough to prove that for $\varphi \in (H_1)''$, $\varphi(f_v) = 0$ for all $v \in H$ implies $\varphi = 0$.

So, let $\varphi \in (H_1)''$ such that $\varphi(f_v) = 0$ for all $v \in H$. Since H_1 is reflexive, there exists $u \in H_1$ such that

$$\varphi(f) = f(u) \quad \forall f \in (H_1)'.$$

Thus, $f_v(u) = 0$ for all $v \in H$, i.e., $\langle u, v \rangle = 0$ for all $v \in H$. Hence, $u = 0$; consequently, $\varphi = 0$. □

Theorem

For $v \in H_{-1}$, *let* (v_n) *in* H *is such that* $\|v - v_n\|_{-1} \to 0$. *Then* $\Phi : H_{-1} \to (H_1)'$ *defined by*

$$\Phi(v) := \lim_{n\to\infty} f_{v_n}, \quad v \in H_{-1},$$

is a surjective linear isometry.

Proof.

Let $\Phi : H_{-1} \to (H_1)'$ be defined by

$$\Phi(v) := \lim_{n\to\infty} f_{v_n}, \quad v \in H_{-1},$$

where (v_n) in H is such that $\|v - v_n\|_{-1} \to 0$. Then Φ is linear isometry and

$$\|\Phi(v)\| = \lim_{n\to\infty} \|f_{v_n}\| = \lim_{n\to\infty} \|v_n\|_{-1} = \|v\|_{-1}, \quad v \in H_{-1}.$$

Thus, Φ is a linear isometry.
Now, surjectivity: Let $f \in (H_1)'$. Since $\{f_v : v \in H\}$ is dense in $(H_1)'$, there exists (v_n) in H such that $\|f - f_{v_n}\| \to 0$. Then (v_n) is a Cauchy sequence in $\tilde{H}$ and

$$f = \lim_{n\to\infty} f_{v_n} = \Phi(v),$$

where $v \in H_{-1}$ is such that $\|v - v_n\|_{-1} \to 0$. □

The Hilbert space H_{-1}

- H_{-1} is a Hilbert space; with inner product structure inherited from $(H_1)'$, i.e.,

For $u, v \in H_{-1}$,

$$\langle u, v\rangle_{-1} := \langle \Phi(u), \Phi(v)\rangle.$$

Recall that, for $f, g \in (H_1)'$,

$$\langle f, g\rangle := \langle u_g, u_f\rangle_{H_1},$$

where u_f is the unique element in H_1 such that

$$\langle v, u_f\rangle_{H_1} = f(v), \quad v \in H_1.$$

- Gelfant triple is a triple of Hilbert spaces.

Illustration

Let Ω be a bounded open subset of $\mathbb{R}^k$. Then $H_0^1(\Omega)$ is a dense subspace of $L^2(\Omega)$.

Given $u \in L^2(\Omega)$, there exists a unique $\tilde{u} \in H_0^1(\Omega)$ such that

$$\langle w, \tilde{u} \rangle_{H_0^1(\Omega)} = f_u(w) := \langle w, u \rangle_{L^2(\Omega)} \quad \forall\, w \in H_0^1(\Omega)$$

i.e., for every $u \in L^2(\Omega)$, there exists a unique $\tilde{u} \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla w(x).\nabla \tilde{u}(x)\, dx = \int_\Omega w(x) u(x)\, dx \quad \forall\, x \in H_0^1(\Omega)$$

Thus, $\tilde{u}$ is the weak solution of

$$-\Delta \tilde{u} = u \quad \text{with} \quad u_{|\partial\Omega} = 0.$$

Examples of Hilbert scales

Example

Let H be a separable Hilbert space and $\{u_n : n \in \mathbb{N}\}$ be an orthonormal basis of H. Let (σ_n) be a sequence of positive real numbers with $\sigma_n \to 0$. For $s \geq 0$, let

$$H_s := \{x \in H : \sum_{n=1}^{\infty} \frac{|\langle x, u_n\rangle|^2}{\sigma_n^{2s}} < \infty\}.$$

Then H_s is a Hilbert space with inner product

$$\langle x, y\rangle_s := \sum_{n=1}^{\infty} \frac{\langle x, u_n\rangle\langle u_n, y\rangle}{\sigma_n^{2s}}.$$

For $s < 0$, H_s is defined via Gelfand triple.

- $\{H_s : s \in \mathbb{R}\}$ is a Hilbert scale.
- For $x \in H$ and $s \geq 0$, $\|x\|_{-s} = \sum_{n=1}^{\infty} \sigma_n^{2s} |\|\langle x, u_n\rangle|^2$.

Example

Let H be a Hilbert space and $L : D(L) \subseteq H \to H$ be a densely defined strictly positive self adjont operator which is also coercive., i.e.,

$$\langle Lx, x\rangle \geq \gamma \|x\|^2 \quad \forall x \in H$$

for some $\gamma > 0$.

Let $X = \cap_{k=1}^{\infty} D(L^k)$. For or $s > 0$, let

$$\langle u, v\rangle_s := \langle L^s u, v\rangle, \quad u \in X,$$

where L^s is defined via spectral representation of L.

- $\langle \cdot, \cdot \rangle_s$ is an inner product on X. (continues...)

$\diamondsuit$

Example

[Continuation] Let H_s be the completion of X with respect to $\langle \cdot, \cdot \rangle_s$.

- H_s is a dense subspace of H as a vector space.
- H_s is continuously embedded in H.
- (H_s, H_0, H_{-s}) is Gelfand triple with $H_0 = H$.
- For $s \leq t$, $H_t \subseteq H_s$ and the inclusion is continuous.

$\{H_s : s \in \mathbb{R}\}$ of Hilbert spaces is called the Hilbert scale generated by L.

- Traditionally, Hilbert scale is defined as in this example.

$\diamondsuit$

Remarks

First example is a special case of the second example:

The Hilbert scale is generated by $L : D(L) \subseteq H \to H$,

$$Lx := \sum_{n=1}^{\infty} \frac{\langle x, u_n \rangle}{\sigma_n} u_n, \quad x \in D(L),$$

where

$$D(L) := \{x \in H : \sum_{n=1}^{\infty} \frac{|\langle x, u_n \rangle|^2}{\sigma_n^{2s}} < \infty\}.$$

Note that

$$\langle Lx, x \rangle = \sum_{n=1}^{\infty} \frac{|\langle x, u_n \rangle|^2}{\sigma_n} \geq \frac{1}{\sigma_0} \|x\|^2,$$

where $\sigma_0 := \sup_n \sigma_n$.

- L is the inverse of the compact operator $A : H \to H$ defined by

$$Ax := \sum_{n=1}^{\infty} \sigma_n \langle x, u_n \rangle u_n, \quad x \in H.$$

Consider the Hilbert scale as in second example, generated by an unbounded operator L:

Let $s > 0$ and $u \in H$. Then

$$f_u(x) = \langle x, u \rangle, \quad u \in H_1$$

and $f_u \in (H_s)'$. By Riesz representation theorem, there exists a unique $\tilde{u} \in H_1$ such that

$$f_u(x) = \langle x, \tilde{u} \rangle_s, \quad x \in H_s.$$

Thus,

$$\langle x, u \rangle = \langle x, L^s \tilde{u} \rangle \quad \forall\, x \in H,$$

i.e., $\tilde{u}$ is the unique element in H_s such that

$$L^s \tilde{u} = u.$$

- Even if $u \in H_s$, the element $\tilde{u}$ need not be equal to u.

Hilbert scales induced by compact operators

Let X and Y be Hilbert spaces and $K : X \to Y$ be a compact operator. The *singular value representation* of K:

$$Kx = \sum_{n=1}^{\infty} \sigma_n \langle x, u_n \rangle v_n, \quad x \in X.$$

- (σ_n) is a seqnece of positive reals with $\sigma_n \to 0$,
- $\{u_n : n \in \mathbb{N}\}$ is an orthonormal basis of $H := N(K)^{\perp}$,
- $\{v_n : n \in \mathbb{N}\}$ is an orthonormal basis of $\overline{R(K)} = N(K^*)^{\perp}$.

Further,

$$(K^*K)^{\frac{1}{2}} x = \sum_{n=1}^{\infty} \sigma_n \langle x, u_n \rangle u_n, \quad x \in X.$$

$$Lx := (K^*K)^{-\frac{1}{2}} x = \sum_{n=1}^{\infty} \frac{\langle x, u_n \rangle}{\sigma_n} u_n, \quad x \in D(L),$$

where $D(L) := \{x \in H : \sum_{n=1}^{\infty} \frac{|\langle x, u_n \rangle|^2}{\sigma_n^2} < \infty\}$.

Theorem

Let $\{H_s : s \in \mathbb{R}\}$ be as in the first example and

$$u_n^{(s)} := \sigma_n^s u_n, \quad s \in \mathbb{R},\ n \in \mathbb{N}.$$

Then

(i) $\{u_n^{(s)} : n \in \mathbb{N}\}$ *is an orthonormal basis of* H_s.

(ii) *For* $s < t$, *the inclusiion map* $\mathcal{I}_{s,t} : H_t \to H_s$ *is a compact embedding.*

Proof.

For $x \in H_s$, we have

$$\langle x, u_j^{(s)} \rangle_s = \sum_{n=1}^{\infty} \frac{\langle x, u_n \rangle \langle u_n, u_j^{(s)} \rangle}{\sigma_n^{2s}} = \sum_{n=1}^{\infty} \sigma_j^s \frac{\langle x, u_n \rangle \langle u_n, u_j \rangle}{\sigma_n^{2s}} = \frac{\langle x, u_j \rangle}{\sigma_j^s}.$$

Hence,

$$\langle u_i^{(s)}, u_j^{(s)} \rangle_s = \frac{\langle u_i^{(s)}, u_j \rangle}{\sigma_j^s} = \frac{\sigma_i^s \langle u_i, u_j \rangle}{\sigma_j^s} = \delta_{ij}$$

and for $x \in H_s$,

$$x = \sum_{n=1}^{\infty} \langle x, u_n \rangle u_n = \sum_{n=1}^{\infty} \sigma_n^s \langle x, u_n^{(s)} \rangle_s \, u_n = \sum_{n=1}^{\infty} \langle x, u_n^{(s)} \rangle_s \, u_n^{(s)}.$$

Hence, $\{u_n^{(s)} : n \in \mathbb{N}\}$ is an orthonormal basis of H_s. □

(continues...)

Continuation.

Also, for $x \in H_t$,

$$x = \sum_{n=1}^{\infty} \langle x, u_n \rangle u_n = \sum_{n=1}^{\infty} \sigma_n^t \langle x, u_n^{(t)} \rangle_t \, u_n = \sum_{n=1}^{\infty} \sigma_n^{t-s} \langle x, u_n^{(t)} \rangle_t \, u_n^{(s)}$$

Thus,

$$\mathcal{I}_{s,t} x = \sum_{n=1}^{\infty} \sigma_n^{t-s} \langle x, u_n^{(t)} \rangle u_n^{(s)}, \quad x \in H_t.$$

Since $\sigma_n^{t-s} \to 0$, the inclusion map $\mathcal{I}_{s,t}$ is a compact operator, and the above representation is its singular value representation.

□

Sobolev scale

For $s \geq 0$, recall that

$$H^s(\mathbb{R}^k) := \{f \in L^2(\mathbb{R}^k) : \int_{\mathbb{R}^k} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 d\xi < \infty\}$$

is a Hilbert space with inner product

$$\langle f, g \rangle_s := \int_{\mathbb{R}^k} (1 + |\xi|^2)^s \hat{f}(\xi) \overline{\hat{g}(\xi)} d\xi$$

and the corresponding norm

$$\|f\|_s := \left[\int_{\mathbb{R}^k} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 d\xi\right]^{1/2}.$$

For $s < 0$, $H^s(\mathbb{R}^k)$ is defined via Gelfand triple.

- $\{H^s(\mathbb{R}^k) : s \in \mathbb{R}\}$ is a Hilbert scale, called the Sobolev scale

Interpolation Inequality in Hilbert Scales

Let $\{H_s\}_{s\in\mathbb{R}}$ be a Hilbert scale and $r < s < t$. The we have

$$H_t \subseteq H_s \subseteq H_r, \quad s = (1-\lambda)r + \lambda t, \quad \lambda := \frac{t-s}{t-r}.$$

and there exist constants $c_{r,s}$ and $c_{s,t}$ such that

$$c_{s,t}\|u\|_t \leq \|u\|_s \leq c_{r,s}\|u\|_r \quad \forall\, u \in H_t.$$

In most of the standard Hilbert scales, we have the **interpolation inequality**:

$$\|u\|_s \leq \|u\|_r^{1-\lambda}\|u\|_t^{\lambda}.$$

Example

Let $\{H_s\}_{s\in\mathbb{R}}$ be the Hilbert scale as in first example.

Let $r < s < t$ and $u \in H_t$. Then

$$s = (1-\lambda)r + \lambda t \quad \text{with} \quad \lambda := \frac{(t-s)}{(t-r)}.$$

We write

$$\sum_{n=1}^{\infty} \frac{|\langle u, u_n\rangle|^2}{\sigma_n^{2s}} = \sum_{n=1}^{\infty} \left[\frac{|\langle u, u_n\rangle|^{2(1-\lambda)}}{\sigma_n^{2(1-\lambda)r}}\right] \left[\frac{|\langle u, u_n\rangle|^{2\lambda}}{\sigma_n^{2\lambda t}}\right].$$

♢

(continues...)

Example

[Continuaton] Applying Hölder's inequality with $p = \frac{1}{1-\lambda}$, $q = \frac{1}{\lambda}$, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{|\langle u, u_n\rangle|^2}{\sigma_n^{2s}} &= \sum_{n=1}^{\infty} \left[\frac{|\langle u, u_n\rangle|^{2(1-\lambda)}}{\sigma_n^{2(1-\lambda)r}}\right] \left[\frac{|\langle u, u_n\rangle|^{2\lambda}}{\sigma_n^{2\lambda t}}\right] \\
&\leq \left\{\sum_{n=1}^{\infty} \frac{|\langle u, u_n\rangle|^2}{\sigma_n^{2r}}\right\}^{1-\lambda} \left\{\sum_{n=1}^{\infty} \frac{|\langle u, u_n\rangle|^2}{\sigma_n^{2t}}\right\}^{\lambda} \\
&= \left\{\|u\|_r^2\right\}^{1-\lambda} \left\{\|u\|_t^2\right\}^{\lambda}.
\end{aligned}$$

Thus, interpolation inequality

$$\|u\|_s \leq \|u\|_r^{1-\lambda} \|u\|_t^{\lambda}$$

holds. ♢

Example

Consider Sobolev scale $\{H^s(\mathbb{R}^k)\}_{s\in\mathbb{R}}$. For $r < s < t$, we have

$$\begin{aligned}
\|f\|_s^2 &= \int_{\mathbb{R}^k} (1+|\xi|^2)^s |\hat{f}(\xi)|^2 d\xi \\
&= \int_{\mathbb{R}^k} [(1+|\xi|^2)^r |\hat{f}(\xi)|^2]^{(1-\lambda)} [(1+|\xi|^2)^t |\hat{f}(\xi)|^2]^{\lambda} d\xi \\
&\leq \left[\int_{\mathbb{R}^k} (1+|\xi|^2)^r |\hat{f}(\xi)|^2 d\xi\right]^{1-\lambda} \left[\int_{\mathbb{R}^k} (1+|\xi|^2)^t |\hat{f}(\xi)|^2\right]^{\lambda} d\xi \\
&= \|f\|_r^{2(1-\lambda)} \|f\|_t^{2\lambda}.
\end{aligned}$$

Thus, interpolation inequality

$$\|f\|_s \leq \|f\|_r^{1-\lambda} \|f\|_t^{\lambda}$$

holds. $\diamondsuit$

Example

Let $\{H_s\}_{s\in\mathbb{R}}$ be the Hilbert scale as in second example. In this case, for $s = (1-\lambda)r + \lambda t$, we have

$$\begin{aligned} \|x\|_s^2 &= \langle L^s x, x\rangle = \int_0^\infty \lambda^s d\langle E_\lambda x, x\rangle \\ &= \int_0^\infty \lambda^{r(1-\lambda)}\lambda^{t\lambda} d\langle E_\lambda x, x\rangle. \end{aligned}$$

Hence, by Hölder's inequality,

$$\begin{aligned} \|x\|_s^2 &\leq \Big(\int_0^\infty \lambda^r d\langle E_\lambda x, x\rangle\Big)^{1-\lambda}\Big(\int_0^\infty \lambda^t d\langle E_\lambda x, x\rangle\Big)^{\lambda} \\ &= \|x\|_r^{2(1-\lambda)}\|x\|_t^{2\lambda}. \end{aligned}$$

Thus,

$$\|x\|_s \leq \|x\|_r^{1-\lambda}\|x\|_t^{\lambda}$$

holds for all $x \in H_t$. ◊

Part II

Ill-Posed Operator Equations

Let X and Y be Banach spaces. For a given $y \in Y$, consider the problem of finding a solution x of the operator equation

$$F(x) = y, \tag{1}$$

where $F : D(F) \subseteq X \to Y$.

According to Hadamard[1], the above problem is said to be **well-posed** if

- for every $y \in Y$ there is a solution x,
- the solution x is unique, and
- the solution depends continuously on the data (y, F).

[1] J. Hadamard, *Lectures on the Cauchy Problem in Linear partial Differential Equations*, Yale University Press, 1923

Definition

If the problem is not a well-posed problem, then it is called an **ill-posed** problem. ♢

Operator theoretic formulation of many of the inverse problems that appear in science and engineering are ill-posed.

Here are two typical examples of ill-posed problems:

1. Fredholm integral equations of the first kind[2]:
   - Computerized tomography,
   - Geophysical prospecting,
   - Image reconstruction problems.
2. Parameter identification problems in PDE[3] [4] [5]:
   - diffraction tomography,
   - impedance tomography,
   - oil reservoir simulation, and
   - under water hydrology.

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[2] Engl, H. W., Hanke, M. and Neubauer, A, *Regularization of Inverse Problems*, Kluwer, Dordrecht, 1993.

[3] R. L. Parker, *Geophysical Inverse Theory*, Princeton University Press, Princeton NJ, 1994.

[4] L. Borcea, J. G. Berryman, and G. C. Papanicolaou, High-contrast impedance tomography, *Inverse Problems*, **12** (1996).

[5] A. J. Devaney, The limited-view problem in diffraction tomography, *Inverse Problems*, **5**(5)(1989).

Example

(Fredholm integral equations of the first kind)

The problem is to solve integral equation

$$\int_\Omega k(s,t)x(t)\,dt = y(s), \quad x \in X,\ s \in \Omega,$$

where $k(\cdot,\cdot)$ is a non-degenerate kernel in $L^2(\Omega \times \Omega)$ and $y \in L^2(\Omega)$.

- The operator $T : L^2(\Omega) \to L^2(\Omega)$ defined by

  $$(Tx)(s) = \int_\Omega k(s,t)x(t)\,dt, \quad x \in X,\ s \in \Omega,$$

  is a *compact operator with non-closed range*.

◇

Thus, the problem of solving such integral equations is ill-posed.

Illustration:

Let $x, y \in L^2[0,1]$ be such that $Tx = y$.
Consider a perturbed data

$$y_n(s) = y(s) + \varepsilon_n(s)$$

with

$$\varepsilon_n(s) := \int_0^1 k(s,t)\sin(n\pi t)dt, \quad s \in [0,1].$$

Then $Tx_n = y_n$ with

$$x_n(t) := x(t) + \sin(n\pi t).$$

- $\|y_n - y\|_2 = \|\varepsilon_n(\cdot)\|_2 \to 0;$
- $\|x_n - x\|_2 = \frac{1}{2}$ for all $n \in \mathbb{N}$.

Therefore, the integral equation is ill-posed.

Proof.

Recall:

> (φ_n) with $\varphi_n(s) := \sqrt{2}\sin(n\pi s)$ is an orthonormal sequence in $L^2[0,1]$.

Hence, for each $s \in [0,1]$,

$$\varepsilon_n(s) = \int_0^1 k(s,t)\sin(n\pi t)dt = \frac{1}{\sqrt{2}}\langle k(s,\cdot), \varphi_n\rangle \to 0,$$

$$|\varepsilon_n(s)|^2 \le \frac{1}{2}\Big(\int_0^1 |k(s,t)|^2 dt\Big),$$

so that

$$\int_0^1 |\varepsilon_n(s)|^2 ds \le \frac{1}{2}\int_0^1 \Big(\int_0^1 |k(s,t)|^2 dt\Big) ds < \infty.$$

□

Continuation.

Thus,

- $|\varepsilon_n(s)|^2 \to 0$ for each $s \in [0, 1]$ and
- $|\varepsilon_n(\cdot)|^2$ is integrable.

Hence, by Dominated Convergence Theorem,

$$\|\varepsilon_n(\cdot)\|_2 = \int_0^1 |\varepsilon_n(s)|^2 ds \to 0 \quad \text{as} \quad n \to \infty.$$

Also,

$$\|x_n - x\|_2 = \int_0^1 \sin^2(n\pi t)dt = \frac{1}{2} \quad \forall\, n \in \mathbb{N}.$$

Thus, we arrive at the situation:

$$\|y_n - y\| \to 0 \quad \text{but} \quad \|x_n - x\| \not\to 0.$$

□

Compact operator equations

Let $T : X \to Y$ be a compact operator between Banach spaces X and Y.

- T infinite rank implies T is not bounded below.
- T not bounded below implies for every sequence (λ_n) with $\lambda_n > 0$, there exists (u_n) in X such that

$$\|Tu_n\| \leq \lambda_n \quad \text{and} \quad \|u_n\| = \frac{1}{\lambda_n} \quad \forall\, n \in \mathbb{N}.$$

In particular, if $Tx = y$ and for $n \in \mathbb{N}$ if

$$y_n = y + Tu_n, \quad x_n = x + u_n,$$

then $Tx_n = y_n$,

$$\|y - y_n\| \leq \lambda_n \quad \text{and} \quad \|x - x_n\| = \frac{1}{\lambda_n}.$$

Illustration

Let X and Y be Hilbert spaces and $T : X \to Y$ be a compact operator of *infinite rank*. Consider its singular value representation:

$$Tx = \sum_{n=1}^{\infty} \sigma_n \langle x, u_n \rangle v_n, \quad x \in X.$$

Let $x \in X$ and $y = Tx$. For $n \in \mathbb{N}$, let

$$y_n = y + \sqrt{\sigma_n} v_n, \quad x_n = x + \frac{1}{\sqrt{\sigma_n}} u_n.$$

Then we have

$$Tx_n = y_n \quad \forall\, n \in \mathbb{N}.$$

Note that, as $n \to \infty$,

$$\|y - y_n\| = \sqrt{\sigma_n} \to 0 \quad \text{but} \quad \|x - x_n\| = \frac{1}{\sqrt{\sigma_n}} \to \infty.$$

Example

(Parameter identification problem in PDE)

Let Ω be a bounded domain in $\mathbb{R}^n$ with *smooth* boundary $q(\cdot) \in L^\infty(\Omega)$ be such that $q(\cdot) \geq c_0$ a.e. for some $c_0 > 0$. Then for every $f \in L^2(\Omega)$, there exists a unique $u \in H^2(\Omega)$ such that

$$-\nabla.(q(x)\nabla u) = f, \qquad u_{|\partial\Omega} = 0.$$

- The map $F : q \mapsto u$ is a nonlinear operator which does not have a continuous inverse.

Thus, the above parameter identification problem in PDE is a nonlinear ill-posed problem. $\diamondsuit$

$$\frac{d}{dt}\left[q(t)\frac{du}{dt}\right] = f(t), \quad 0 < t < 1,$$

where $f \in L^2(0,1)$. Note that

$$u(t) = \int_0^t \left[\frac{1}{q(\tau)}\int_0^\tau f(s)ds\right] d\tau.$$

Thus, the problem is same as that of solving the equation

$$F(q)(t) := \int_0^t \left[\frac{1}{q(\tau)}\int_0^\tau f(s)ds\right] d\tau = u(t),$$

Clearly, this equation is nonlinear.

Note also that

$$q(t) = \frac{1}{u'(t)} \int_0^t f(s)ds.$$

Suppose $u(t)$ is perturbed to $\tilde{u}(t)$, say

$$\tilde{u}(t) = u(t) + \varepsilon(t).$$

Suppose $\tilde{q}(t)$ is the corresponding solution. Then we have

$$\begin{aligned} q(t) - \tilde{q}(t) &= \left[\frac{1}{u'(t)} - \frac{1}{u'(t) + \varepsilon'(t)}\right] \int_0^t f(s)ds \\ &= \frac{1}{u'(t)} \left[\frac{\varepsilon'(t)}{u'(t) + \varepsilon'(t)}\right] \int_0^t f(s)ds. \end{aligned}$$

Hence,

$$\varepsilon'(t) \approx \infty \quad \Longrightarrow \quad |q(t) - \tilde{q}(t)| \approx |q(t)|.$$

There can be perturbations $\varepsilon(t)$ such that

$$\varepsilon(t) \approx 0 \quad \text{but} \quad \varepsilon'(t) \approx \infty.$$

For example, for large n,

$$\varepsilon_n(t) = (1/n)\sin(n^2 x) \approx 0 \quad \text{but} \quad \varepsilon_n'(t) = n\cos(n^2 x) \approx \infty.$$

Thus, the problem is ill-posed.

Regularization

For an ill-posed problem with an approximate data $(\tilde{y}, \tilde{F})$ in place of (y, F), one looks for a family $\{\tilde{x}_\alpha\}_{\alpha>0}$ of approximate solutions such that

- each $\tilde{x}_\alpha$ is a solution of a well-posed problem and
- $\alpha := \alpha(\tilde{y}, \tilde{F})$ is chosen in such a way that

$$\tilde{x}_\alpha \to x \quad \text{as} \quad (\tilde{y}, \tilde{F}) \to (y, F).$$

The procedure of finding such a stable approximate solution is called a **regularization method**.

Tikhonov regularization

Consider a linear ill-posed problem:

$$Tx = y,$$

where $T : X \to Y$ is a bounded linear operator between Hilbert spaces X and Y with nonclosed range $R(T)$.

We may recall:

- $x_0 \in X$ is an *LRN-solution* if

$$\|Tx_0 - y\| = \inf_{x \in X} \|Tx - y\|;$$

- An LRN-solution x_0 exists if and only if $y \in R(T) + R(T)^{\perp}$, and in that case

$$T^* T x_0 = T^* y.$$

- If $y \in R(T) + R(T)^{\perp}$, then there exists a unique LRN-solution $x^{\dagger} := T^{\dagger} y$ of minimal norm.

- One looks for stable approximations for the the minimum norm LRN- solution

$$x^\dagger := T^\dagger y, \quad y \in D(T^\dagger) := R(T) + R(T)^\perp.$$

In *Tikhonov regularization*, the regularized solution is the unique minimizer of

$$x \mapsto \|Tx - \tilde{y}\|^2 + \alpha\|x\|^2,$$

equivalently, the unique solution of the well-posed operator equation

$$(T^*T + \alpha I)\tilde{x}_\alpha = T^*\tilde{y}.$$

It is known[6]:

- If $y \in D(T^\dagger)$, and $\|\tilde{y} - y\| \leq \delta$ for some $\delta > 0$, then the best possible error estimate is

$$\|x^\dagger - \tilde{x}_\alpha\| = O(\delta^{2/3}).$$

- The above estimate is order optimal for the source set

$$\{x \in X : x = (T^*T)u : \|u\| \leq \rho\}$$

- It is attained by an a priori choice of α, namely, $\alpha \sim \delta^{2/3}$ or by the a posteriori choice of Arcangeli's method[7]

$$\|T\tilde{x}_\alpha - \tilde{y}\| = \frac{\delta}{\sqrt{\alpha}}.$$

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[6] M.T. Nair, *Linear Operator Equations: Approximation and Regularization*, World Scientific, Singapore, New York, 2009.

[7] M.T. Nair, A generalization of Arcangeli's method for ill-posed problems leading to optimal rates, *Integral Equations and Operator Theory*, **15** (1992), 1042-1046.

In order to improve the error estimate, Natterer[8] suggested a modification using a Hilbert scale $\{H_s : s \in \mathbb{R}\}$ to obtain approximations for the LRN-solution which minimizes the function

$$x \mapsto \|x\|_s.$$

It is assumed that the interpolation inequality

$$\|u\|_s \leq \|u\|_r^{1-\lambda} \|u\|_t^{\lambda}, \quad s = (1-\lambda)r + \lambda t$$

holds.

[8] F. Natterer, Error bounds for Tikhonov regularization in Hilbert scales, *Applicable Analysis*, **18** (1984), 29–37.

In this modification, the regularized solution is the the minimizer $\tilde{x}_{\alpha,s}$ of

$$x \mapsto \|Tx - \tilde{y}\|^2 + \alpha\|x\|_s^2.$$

Natterer showed:

- If T satisfies

$$\|Tx\| \geq c\|x\|_{-a} \quad \forall x \in X$$

  for some $a > 0$ and $c > 0$,
- $\hat{x} \in H_t$ where $0 \leq t \leq 2s + a$, and
- $\alpha \sim \delta^{\frac{2(a+s)}{a+t}}$,

then

$$\|\tilde{x}_{\alpha,s} - \hat{x}\| = O(\delta^{\frac{t}{t+a}}). \quad (2)$$

Thus,

higher smoothness requirement on $\hat{x}$ and with higher level of regularization gives higher order of convergence.

Using unbounded stabilizing operators

Another approach is to look for an approximation of the LRN-solution which minimizes the function

$$x \mapsto \|Lx\|,$$

where $L : D(L) \subseteq X \to X$ is a closed densely defined operator. It is known[9] that such an LRN-solution exists whenever

$$y \in R(T|_{D(L)}) + R(T)^{\perp}.$$

Accordingly, one finds the unique minimizer $\tilde{x}_\alpha$ of the function

$$x \mapsto \|Tx - \tilde{y}\|^2 + \alpha\|Lx\|^2, \quad x \in D(L),$$

[9] M.T. Nair, M. Hegland and R.S. Anderssen, The trade-off between regularity and stability in Tikhonov regularization, *Math. Comp.*, **66** (1997), 193–206

Existence and uniqueness of the regularized solutions $x_\alpha(\tilde{y})$ are ensured by assuming the completion condition[10 11 12]:

$$\|Tx\|^2 + \|Lx\|^2 \geq \gamma \|x\|^2 \quad \forall x \in D(L), \tag{3}$$

- The condition $(*)$ is satisfied, if for example, L is bounded below, which is the case for many of the differential operators that appear in applications.
- $L = I$ corresponds to ordinary Tikhonov regularization.

[10] M.T. Nair, M. Hegland and R.S. Anderssen, The trade-off between regularity and stability in Tikhonov regularization, *Math. Comp.*, **66** (1997), 193–206

[11] J. Locker and P.M. Prenter, Regularization with differential operators, *Math. Anal. Appl.*, **74** (1980) , 504–529.

[12] M. Hanke

Also, condition (3) ensures

- $\tilde{x}_\alpha$ is the unique solution of the well-posed equation

$$(T^*T + \alpha L^*L)x = T^*\tilde{y},$$

- $y \in R(T_{|_{D(L)}}) + R(T)^\perp$ implies

$$x_\alpha \to \hat{x} \quad \text{as} \quad \alpha \to 0,$$

where $\hat{x}$ is the unique LRN-solution which minimizes $x \mapsto \|Lx\|$.

Improved error estimates under source conditions

Suppose we have the perturbed data y^δ with

$$\|y - y^\delta\| \le \delta,$$

and let x_α^δ be the corresponding regularized solution, i.e.,

$$(T^*T + \alpha L^*L)x_\alpha^\delta = T^*y^\delta.$$

Required to choose the regularization parameter $\alpha := \alpha(\delta, y^\delta)$ appropriately so that

$$x_\alpha^\delta \to \hat{x} \quad \textit{as} \quad \delta \to 0.$$

In order to obtain error estimates, it is necessary to impose some smoothness assumptions on $\hat{x}$, by requiring it to belong to certain *source set*.

This aspect has been considered extensively in the literature in recent years by assuming that the operators T, L are associated with a Hilbert scale $\{X_s\}_{s\in\mathbb{R}}$ in an appropriate manner[13] [14] [15].

One such relation is as in the following.

[13] F. Natterer, Error bounds for Tikhonov regularization in Hilbert scales, *Applicable Analysis*, **18** (1984), 29–37.

[14] M.T. Nair, M, E. Schock and U. Tautenhahn, U, Morozov's discrepancy principle under general source conditions, *J. Anal. Anw.* , **22** (2003), 199–214.

[15] M.T. Nair, M, S. Pereverzyev and U. Tautenhahn, Regularization in Hilbert scales under general smoothing conditions, *Inverse Problems*, **21** (2003), 1851-1869.

Hilbert-scale conditions:

(i) There exists $a > 0$, $c > 0$ such that

$$\|Tx\| \geq c\|x\|_{-a} \quad \forall x \in X. \tag{4}$$

(ii) There exists $b \geq 0$, $d > 0$ such that $D(L) \subseteq X_b$ and

$$\|Lx\| \geq d\|x\|_b \quad \forall x \in D(L). \tag{5}$$

The following result is proved in the paper[16].

[16] M.T. Nair, On Morozov's method Tikhonov regularization as an optimal order yielding algorithm, *Zeit. Analysis und ihre Anwendungen*, **18** (no.1) (1999) 37-46.

Theorem (1)

If the Hilbert scale conditions (4) and (5) are satisfied and if $\hat{x}$ belongs to the source set

$$M_\rho = \{x \in D(L) : \|Lx\| \leq \rho\} \tag{6}$$

for some $\rho > 0$, α is chosen according to the Morozov discrepancy principle

$$c_1\delta \leq \|Tx_\alpha^\delta - y^\delta\| \leq c_0\delta \tag{7}$$

with $c_0, c_1 \geq 1$, then

$$\|\hat{x} - \tilde{x}_\alpha\| \leq 2\left(\frac{\rho}{d}\right)^{\frac{a}{a+b}}\left(\frac{\delta}{c}\right)^{\frac{b}{a+b}}. \tag{8}$$

The estimate in (8) corresponds to the estimate (2) obtained by Natterer for the case $t = s = b$.

For obtaining further improved estimate, two more source sets are considered in Nair[17]:

$$\widetilde{M}_\rho = \{x \in D(L) : \|L^*Lx\| \le \rho\}, \tag{9}$$

$$M_{\rho,\varphi} := \{x \in D(L^*L) : \; L^*Lx = [\varphi(T^*T)]^{1/2}u, \; \|u\| \le \rho\}. \tag{10}$$

- $\varphi : [0,\infty) \to [0,\infty)$ is a strictly monotonically increasing continuous function such that $\lim_{\lambda\to 0} \varphi(\lambda) = 0$.

In regularization theory, such functions are called index functions.

[17] M.T. Nair, On improving estimates for Tikhonov regularization using an unbounded operator, *J. Analysis*, **14** (2006), 143–157.

Theorem (2)

Suppose the Hilbert scale conditions (4) and (5) are satisfied and α is chosen according to the Morozov discrepancy principle (7).

(i) *If $\hat{x}$ belongs to the source set $\widetilde{M}_\rho$ defined in (9), then*

$$\|\hat{x} - \tilde{x}_\alpha\| \leq 2\left(\frac{\rho}{d^2}\right)^{\frac{a}{2a+b}} \left(\frac{\delta}{c}\right)^{\frac{2b}{a+2b}}.$$

(ii) *If $\hat{x}$ belongs to the source set $M_{\rho,\varphi}$ defined in (10), then*

$$\|\hat{x} - \tilde{x}_\alpha\| = (1 + c_0)\left(\frac{\rho}{d^2}\right)^{\frac{a}{a+2b}} \left(\frac{\delta}{c}\right)^{\frac{2b}{a+2b}} \left[\psi_p^{-1}\left(\varepsilon_\delta^2\right)\right]^{\frac{a}{2(a+2b)}}$$

where

$$p = \frac{a}{a+2b}, \quad \varepsilon_\delta := c\left(\frac{d^2\delta}{c\rho}\right)^{\frac{a}{a+2b}}, \quad \psi_p(\lambda) := \lambda^{1/p}\varphi^{-1}(\lambda).$$

In a recent paper[18] a unified approach is adopted by by replacing the Hilbert scale conditions (4) and (5) by a single condition involving T and L:

θ-condition: There exist $\eta > 0$ and $0 \leq \theta < 1$ such that

$$\eta\|x\| \leq \|Tx\|^{\theta}\|Lx\|^{1-\theta} \quad \forall x \in D(L). \tag{11}$$

[18] M.T. Nair, A Unified Treatment for Tikhonov Regularization Using a General Stabilizing Operator, *Analysis and Applications*, (2013)

In a recent paper[18] a unified approach is adopted by by replacing the Hilbert scale conditions (4) and (5) by a single condition involving T and L:

θ-condition: There exist $\eta > 0$ and $0 \leq \theta < 1$ such that

$$\eta\|x\| \leq \|Tx\|^{\theta}\|Lx\|^{1-\theta} \quad \forall x \in D(L). \tag{11}$$

Observe:

- $\theta = 0$ corresponds to L being bounded below so that $R(L)$ is closed and $L^{-1} : R(L) \to X$ is a bounded operator. This case also include the choice $L = I$, the identity operator.
- $\theta = 1$ is excluded, as it would imply that T has a continuous inverse.

[18] M.T. Nair, A Unified Treatment for Tikhonov Regularization Using a General Stabilizing Operator, *Analysis and Applications*, (2013)

Theorem

1. *θ-condition:*

$$\eta\|x\| \leq \|Tx\|^{\theta}\|Lx\|^{1-\theta}$$

implies the completion condition (3):

$$\|Tx\|^2 + \|Lx\|^2 \geq \gamma\|x\|^2$$

with $\gamma = \eta^2$;

2. *Hilbert scales conditions (4) and (5):*

$$\|Tx\| \geq c\|x\|_{-a}, \quad \|Lx\| \geq d\|x\|_{b}.$$

imply θ-condition:

$$\eta\|x\| \leq \|Tx\|^{\theta}\|Lx\|^{1-\theta}$$

with $\theta = \dfrac{b}{a+b}$ and $\eta = c^{\theta}d^{1-\theta}$.

Proof.

1 Suppose θ-condition is satisfied. From the relation $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ with

$$a = \|Tx\|^{2\theta}2, \quad b = \|Lx\|^{2(1-\theta)}, \quad p = \frac{1}{\theta}, \quad q = \frac{1}{1-\theta},$$

we obtain

$$\|Tx\|^{2\theta}\|Lx\|^{2(1-\theta)} \leq \theta\|Tx\|^2 + (1-\theta)\|Lx\|^2.$$

Thus,

$$\eta^2\|x\|^2 \leq \|Tx\|^2 + \|Lx\|^2.$$

□

Proof.

2 Suppose Hilbert scales conditions are satisfied, i.e.,

$$\|Tx\| \geq c\|x\|_{-a}, \quad \|Lx\| \geq d\|x\|_b.$$

Then

$$\|x\|_0 \leq \|x\|_{-a}^{\theta}\|x\|_b^{1-\theta},$$

where θ is sauch that $0 = (1-\theta)(-a) + \theta b$, i.e., $\theta = \dfrac{a}{a+b}$.

Thus,

$$\|x\|_0 \leq (\|Tx\|/c)^{\theta}(\|Lx\|/d)^{1-\theta},$$

so that

$$\eta\|x\| \leq \|Tx\|^{\theta}\|Lx\|^{1-\theta}$$

with $\eta = c^{\theta}d^{1-\theta}$.

□

The following theorem[19] unifies results in the setting of general unbounded stabilizing operator as well as for Hilbert-scale and Hilbert-scale-free settings.

[19] M.T. Nair, A Unified Treatment for Tikhonov Regularization Using a General Stabilizing Operator, *Analysis and Applications*, (2013)

Theorem (3)

Under the θ-condition (11) and the discrepancy principle (7), the following hold.

(i) *If $\hat{x} \in M_\rho$, then*

$$\|\hat{x} - x^\delta_{\alpha_\delta}\| \leq 2\eta^{-1}\rho^{1-\theta}\delta^\theta.$$

(ii) *If $\hat{x} \in \widetilde{M}_\rho$, then*

$$\|\hat{x} - x^\delta_{\alpha_\delta}\| \leq (1 + c_0)\left(\frac{1}{\eta^2}\right)^{\frac{1}{1+\theta}} \rho^{\frac{1-\theta}{1+\theta}}\delta^{\frac{2\theta}{1+\theta}}.$$

(iii) *If $\hat{x} \in M_{\rho,\varphi}$ and $\delta^2 \leq \gamma_1^2\varphi(1)$ where $\gamma_1 = 4\rho/\gamma$, then*

$$\|\hat{x} - x^\delta_{\alpha_\delta}\| \leq (1 + c_0)\left(\frac{1}{\eta^2}\right)^{\frac{1}{1+\theta}} \rho^{\frac{1-\theta}{1+\theta}}\delta^{\frac{2\theta}{1+\theta}}[\psi_p^{-1}(\varepsilon_\delta^2)]^{1/2p},$$

where $p := \frac{1+\theta}{1-\theta}, \quad \varepsilon_\delta := \eta^{\frac{2}{1+\theta}}\left(\frac{\delta}{\rho}\right)^{1/p}.$

Remark: The results in Theorems (1) and (2) are recovered from Theorem (3), (i) and (ii), respectively, by taking

$$\theta = \frac{b}{a+b}, \quad \eta = c^{\frac{b}{a+b}} d^{\frac{a}{a+b}}.$$

In the ordinary Tikhonov regularization, i.e., for the case of $L = I$, equivalently, $\theta = 0$ in part (iii) of the above theorem, we have $p = 1$ and $\varepsilon_\delta = \eta^2 \left(\frac{\delta}{\rho}\right)$. Hence, the estimate reduces to

$$\|\hat{x} - x_{\alpha_\delta}^{\delta}\| \leq (1 + c_0) \left(\frac{1}{\eta^2}\right) \sqrt{\psi_1^{-1}(\eta^2\delta^2/\rho^2)}.$$

Thus, we recover the error estimate for ordinary Tikhonov regularization under the general source condition[20].

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[20] M.T. Nair, M, E. Schock and U. Tautenhahn, U, Morozov's discrepancy principle under general source conditions, *J. Anal. Anw.* , **22** (2003), 199–214.

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Thank you for your attention!