Syllabus for the Comprehensive Examination

Compulsory Topics:

1. Linear Algebra:

Systems of Linear Equations, Matrices and Elementary Row Operations, Row-Reduced Echelon Matrices .

Vector Spaces, Subspaces, Bases and Dimension, Ordered basis and coordinates.

Linear transformations, Rank-Nullity Theorem, The algebra of linear transformations, Isomorphism, Matrix representation of linear transformations, Linear Functionals, Annihilator, Double dual, Transpose of a linear transformation .

Characteristic Values and Characteristic Vectors of linear transformations, Diagonalizability, Minimal polynomial of a linear transformation, Cayley-Hamilton Theorem, Invariant Subspaces, Direct-sum decompositions, Invariant Direct sums, The primary decomposition theorem, Cyclic subspaces and annihilators, Cyclic decomposition, rational, Jordan forms.

Inner Product Spaces, Orthonormal Basis, Gram-Schmidt Theorem.

2. Real Analysis:

Real number system and its order completeness, sequences and series of real numbers.

Metric spaces: Basic concepts, continuous functions, completeness, contraction mapping theorem, connectedness, Intermediate Value Theorem, Compactness, Heine-Borel Theorem.

Differentiation, Taylor's theorem, Riemann Integral, Improper integrals.

Sequences and series of functions, Uniform convergence, power series, Fourier series, Weierstrass approximation theorem, equicontinuity, Arzela-Ascoli theorem.

Optional Topics: (Any two of the following)

Algebra:

Review of basic Group Theory, Group Actions, Kernel and Stabilizer of Group Actions, Transitive Group Action, Cayley's Theorem, The Class Equation, Sylow's Theorems, Direct Products, Structure Theorem for Finite Abelian Groups, Free Groups.

Review of basic Ring Theory, Properties of Ideals, Prime and Maximal Ideals, Chinese Reminder Theorem, Euclidean Domain, Euclidean Algorithm, Principal Ideal Domain, Euclidean Domain is a Principal Ideal Domain. UFD, PID is UFD, Polynomial Rings over fields and rings, Eisenstein Crieterion for Irreducibility.

Complex Analysis:

Topology of the complex plane, Riemann sphere, limits, continuity and differentiability. Analytic functions, harmonic functions and multi-valued functions.

Convergence of numerical series. Radius of convergence of power series, and power series as an analytic function. Laurent series.

Cauchy's integral theorem, Cauchy integral formula, Morera's theorem, Taylor`s theorem, Laurent's theorem, Liouville's theorem, Schwarz lemma; Maximum Modulus Principle.

Conformal mappings, linear fractional transformations. Classification of singularities, Cauchy's residue theory and evaluation of real integrals.

Differential Equations:

Existence-Uniqueness: Review of exact equations of first order, The method of successive approximations, Lipschitz condition, Convergence of successive approximations, Existence and Uniqueness of solutions for first order initial value problem, Non-local existence of solutions, Existence and uniqueness of solutions to systems, Existence and uniqueness for linear systems, Equations of order n.

Second Order Equations: General solution of homogeneous equations, Non-homogeneous equations, Wronskian, Method of variation of parameters, Sturm comparison theorem, Sturm separation theorem, Boundary value problems, Green's functions, Sturm-Liouville problems.

Series Solution of Second Order Linear Equations: ordinary points, regular singular points, Legendre polynomials and properties, Bessel functions and properties.

Systems of Differential Equations: Algebraic properties of solutions of linear systems, The eigenvalue-eigenvector method of finding solutions, Complex eigenvalues, Equal eigenvalues, Fundamental matrix solutions, Matrix exponential, Nonhomogeneous equations, Variation of parameters.

Discrete Mathematics:

Set Theory and Logic: Deductive Proof techniques, Axiom of Choice, Zorn's Lemma, Cardinality, Schroder-Bernstein Theorem, Countability and uncountability, uncountability of R, Cantor's theorem, Cardinal Arithmetic.

Combinatorics: Numbers and Counting, Partitions and Permutations, Principle of Inclusion and Exclusion, Pigeon Hole Principle, Recurrence Relations, Generating Functions.

Graphs: Basics of Graphs, Relations and Digraphs, Tress, Connectivity, Traversability, Planarity.

Functional Analysis:

Normed linear space; Banach spaces and basic properties: Heine-Borel theorem, Riesz lemma and best approximation property: Inner product space and projection theorem; Orthonormal bases; Bessel inequality and Parseval's formula; Riesz-Fischer theorem.

Bounded operators and basic properties; Space of bounded operators and dual space; Riesz representation theorem; Adjoint of operators on a Hilbert space; Examples of unbounded operators; Convergence of sequence of operators.

Hahn-Banach Extension theorem;Uniform boundedness principle; Closed graph theorem and open mapping theorem. Some applications.

Invertibility of operators; Spectrum of an operator

Numerical Analysis:

Norms of vectors and matrices, liner systems: direct and iterative schemes, ill conditioning and convergence analysis; Numerical Schemes for non-linear systems, Regression, Numerical solution of differential equations: Single step and multi-step methods, order, consistency, stability and convergence analysis, stiff equtations, two point boundary value problems: Shooting and finite difference methods.

Probability & Statistics:

Probability measure, probability space, construction of Lebesgue measure, extension theorems, limit of events, Borel-Cantelli lemma. Random variables, Random vectors, distributions, multidimensional distributions, independence. Expectation, change of variable theorem, Moment generating function and characteristics functions, inversion and uniqueness theorems, continuity theorems, Sequences of random variables, modes of convergence. Weak and strong laws of large numbers, central limit theorem. Radon-Nikodym theorem, definition and properties of conditional expectation, conditional distributions.

Distribution of functions of random variables, Order Statistics. Estimation - loss function, risk, minimum risk unbiased estimators, maximum likelihood estimation, method of moments, Bayes estimation. Sufficient Statistics, completeness, Basu's Theorem, exponential families, invariance and maximal invariant statistics. Testing of Hypotheses - parametric and non-parametric problems, examples with data analytic applications. Confidence Intervals.

Topology:

Topological Spaces, Basis for a topology, Subspace topology, Closed sets and Limit points, Continuous Functions, Product Topology, Quotient Topology.

Connected spaces, Connected subspaces of the Real line, Components and Local Connectedness, Path connectedness, Compact spaces, Limit point compactness, Local compactness.

The Countability and Separation axioms, The Urysohn Lemma, The Urysohn Metrization Theorem, The Tietze Extension Theorem, Tychonoff Theorem.