TOPICS IN FOURIER ANALYSIS-III

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1. Fourier transform: Basic properties

Definition 1.1. Let $d \in \mathbb{N}$ and $f \in L^1(\mathbb{R}^d)$. The Fourier transform of $f$ is the function $\hat{f} : \mathbb{R}^d \to \mathbb{C}$ defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-ix \cdot \xi} dx, \quad \xi \in \mathbb{R}^d.$$ 

Here, $x \cdot \xi$ denotes the dot product of $x, \xi \in \mathbb{R}^d$, i.e., if $x = (x_1, \ldots, x_d), \xi = (\xi_1, \ldots, \xi_d)$ are in $\mathbb{R}^d$, then

$$x \cdot \xi := x_1 \xi_1 + \cdots + x_d \xi_d.$$

Note that the above integral exists for every $\xi \in \mathbb{R}^d$, and hence the function $\hat{f}$ is well-defined.

Theorem 1.2. The map $f \mapsto \hat{f}$ is a bounded linear operator\footnote{Here, $B(\mathbb{R}^d)$ denotes the space of all bounded complex valued functions with sup-norm. This space is a Banach space.} from $L^1(\mathbb{R}^d)$ to $B(\mathbb{R}^d)$.

Proof. Clearly

$$\|\hat{f}\|_\infty \leq \|f\|_1 \quad \text{for all } f \in L^1(\mathbb{R}^d).$$

Hence, $\hat{f} \in B(\mathbb{R}^d)$ for all $f \in L^1(\mathbb{R}^d)$ and $\|\hat{f}\|_\infty \leq \|f\|_1$. Also $f \mapsto \hat{f}$ is a linear operator $L^1(\mathbb{R})$ to $B(\mathbb{R})$. \hfill \Box

Example 1.3. Let $d = 1$ and $f = \chi_{(0,1)}$. Then

$$\hat{f}(\xi) = \int_{-1}^{1} e^{-ix \xi} dx = \begin{cases} 2, & \xi = 0, \\ 2 \sin \xi, & \xi \neq 0. \end{cases}$$

Example 1.4. Let $d = 1$ and $f(x) = e^{-x^2/2}$. Then

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-(x^2/2 + ix \xi)} dx = e^{-\xi^2/2} \int_{-\infty}^{\infty} e^{-(x+ix)^2/2} dx = e^{-\xi^2/2} \int_{-\infty}^{\infty} e^{-x^2/2} dx = \sqrt{2\pi} e^{-\xi^2/2}.$$

More generally, if $f(x) = e^{-|x|^2/2}, x \in \mathbb{R}^d$, then

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-|x|^2/2 + ix \xi} dx = (2\pi)^{d/2} e^{-|\xi|^2/2}.$$

In the above examples we see that $\hat{f}$ is uniformly continuous and $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$. This is, in fact, true for any $f \in L^1(\mathbb{R}^d)$.

Theorem 1.5. Let $f \in L^1(\mathbb{R}^d)$. Then $\hat{f}$ is uniformly continuous.
Proof. For $\xi, h \in \mathbb{R}^d$, we have
\[
\hat{f}(\xi + h) - \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)[e^{-ix(\xi + h)} - e^{-ix\xi}]dx = \int_{\mathbb{R}^d} f(x)e^{-ix\xi}[e^{-ixh} - 1]dx
\]
Thus,
\[
|\hat{f}(\xi + h) - \hat{f}(\xi)| \leq \int_{\mathbb{R}^d} |f(x)| |e^{-ixh} - 1|dx.
\]
Note that
\[
|f(x)||e^{-ixh} - 1| \to 0 \text{ as } |h| \to 0,
\]
with $f \in L^1(\mathbb{R}^d)$. Hence, by DCT,
\[
\int_{\mathbb{R}^d} |f(x)| |e^{-ixh} - 1|dx \to 0
\]
as $|h| \to 0$. Hence, for every $\varepsilon > 0$, there exists $\delta > 0$ such that
\[
|\hat{f}(\xi + h) - \hat{f}(\xi)| \leq \int_{\mathbb{R}^d} |f(x)| |e^{-ixh} - 1|dx < \varepsilon
\]
for every $h$ with $|h| < \delta$. Thus, for every $\varepsilon > 0$, there exists $\delta > 0$ such that
\[
|\hat{f}(\xi + h) - \hat{f}(\xi)| < \varepsilon
\]
for every $h$ with $|h| < \delta$ and for every $\xi \in \mathbb{R}^d$. Thus, $\hat{f}$ is uniformly continuous. \(\square\)

Combining Theorems 1.2 and 1.5, we have

**Theorem 1.6.** The map $f \mapsto \hat{f}$ is a bounded linear operator\(^2\) from $L^1(\mathbb{R}^d)$ to $C_b(\mathbb{R}^d)$ with norm atmost 1.

In fact, we have

**Theorem 1.7.** Let $f \in L^1(\mathbb{R}^d)$. Then
\[
\hat{f}(\xi) \to 0 \text{ as } |\xi| \to \infty.
\]
In particular\(^3\), $\hat{f} \in C_0(\mathbb{R}^d)$ for every $f \in L^1(\mathbb{R}^d)$.

For its proof we shall make use of the following lemma.

---

\(^2\) $C_b(\mathbb{R}^d)$ denotes the space of all bounded, continuous complex valued functions on $\mathbb{R}$ with sup-norm. This space is a Banach space.

\(^3\) $C_0(\mathbb{R}^d)$ denotes the space of all continuous functions $h : \mathbb{R}^d \to \mathbb{C}$ such that $|h(x)| \to 0$ as $|x| \to \infty$. This space is a Banach space.
Lemma 1.8. Let $1 \leq p < \infty$ and $f \in L^p(\mathbb{R}^d)$. Then
\[ \lim_{y \to 0} \int_{\mathbb{R}^d} |f(x) - f(x)|^p dx = 0. \]
In particular, if we define $f_y(x) = f(x - y)$ for $x \in \mathbb{R}^d$, then the map $y \mapsto f_y$ from $\mathbb{R}^d$ to $L^p(\mathbb{R}^d)$ is continuous.

Proof. For $y \in \mathbb{R}^d$ and $f : \mathbb{R}^d \to \mathbb{C}$, let $f_y(x) := f(x - y)$, $x \in \mathbb{R}^d$. Thus, we have to prove that $\|f - f_y\|_p \to 0$ as $y \to 0$ for $f \in L^p(\mathbb{R}^d)$. So, let $f \in L^p(\mathbb{R}^d)$ and $\varepsilon > 0$. By denseness of $C_c(\mathbb{R}^d)$ in $L^p(\mathbb{R}^d)$, there exists $g \in C_c(\mathbb{R}^d)$ such that $\|f - g\|_p < \varepsilon$. Then, we also have $\|f_y - g\|_p < \varepsilon$. Hence,
\[ \|f - f_y\|_p \leq \|f - g\|_p + \|g - g_y\|_p + \|g_y - f_y\|_p < 2\varepsilon + \|g - g_y\|_p. \]
Now, by the uniform continuity of $g$, there exists $\delta > 0$ such that
\[ |g(x) - g(x - y)| < \varepsilon \]
for all $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$ with $|y| < \delta$. Also, there exists a parallelepiped $K \subset \mathbb{R}^d$ such that $g(x) = 0$, $g(x - y) = 0$ for every $x \notin K$ and $|y| < \delta$. Hence,
\[ \|g - g_y\|_p^p = \int_K |g(x) - g(x - y)|^p dx \leq \varepsilon^p m(K) \]
for every $y$ with $|y| < \delta$. Thus,
\[ \|f - f_y\|_p \leq 2\varepsilon + \|m(K)\|^{1/p} \varepsilon \]
so that $\lim_{y \to 0} \|f - f_y\|_p = 0$. Also, for $y, y_0 \in \mathbb{R}^d$,
\[ \|f_y - f_{y0}\|_p = \|f - f_{y-y0}\|_p \to 0 \quad \text{as} \quad y \to y_0. \]
Hence, $y \mapsto f_y$ is continuous on $\mathbb{R}^d$. \hfill \Box

Proof of Theorem 1.7. We observe that for every $\xi \neq 0$,
\[ \int_{\mathbb{R}^d} f(x - \frac{\pi \xi}{|\xi|^2}) e^{-ix \cdot \xi} dx = \int_{\mathbb{R}^d} f(y) e^{-i(y + \pi \xi/|\xi|^2) \cdot \xi} dx = \int_{\mathbb{R}^d} f(y) e^{-iy \cdot \xi} e^{-i\pi dx} = -\hat{f}(\xi). \]
Thus,
\[ \int_{\mathbb{R}^d} f(x - \frac{\pi \xi}{|\xi|^2}) e^{-ix \cdot \xi} dx = -\hat{f}(\xi). \]
Hence,
\[ \hat{f}(\xi) = \frac{1}{2} \int_{\mathbb{R}^d} [f(x) - f(x - \frac{\pi \xi}{|\xi|^2})] e^{-ix \cdot \xi} dx \]
so that
\[ |\hat{f}(\xi)| \leq \frac{1}{2} \int_{\mathbb{R}^d} |f(x) - f \left( x - \frac{\pi \xi}{|\xi|^2} \right) | e^{-ix \cdot \xi} dx = \| f - f_{\pi \xi/|\xi|^2} \|_1. \]
By Lemma 1.8, $\| f - f_{\pi \xi/|\xi|^2} \|_1 \to 0$ as $|\xi| \to \infty$. Thus, $|\hat{f}(\xi)| \to 0$ as $|\xi| \to \infty$. \hfill \Box
Combining Theorems 1.2 and 1.7 we obtain:

**Theorem 1.9.** The map $f \mapsto \hat{f}$ is a bounded linear operator from $L^1(\mathbb{R}^d)$ to $C_0(\mathbb{R}^d)$.

**Theorem 1.10.** Suppose $f \in L^1(\mathbb{R})$ is differentiable with $f' \in L^1(\mathbb{R})$. Then

$$\hat{f}'(\xi) = (i\xi) \hat{f}(\xi), \quad \xi \in \mathbb{R}.$$  

For its proof, we shall make use of the following lemma.

**Lemma 1.11.** If $f \in L^1(\mathbb{R})$ such that $f' \in L^1(\mathbb{R})$, then $f(x) \to 0$ as $x \to \pm \infty$.

**Proof.** Observe that for any $a > 0$,

$$f(a) = f(0) + \int_0^a f'(x)dx.$$  

Hence, $\lim_{a \to \infty} f(a)$ exists, since $f' \in L^1(\mathbb{R})$. Now, suppose that $\lim_{a \to \infty} f(a) = \beta \neq 0$. Then there exists $\alpha > 0$ such that $|f(x)| > |\beta|/2$ for all $x \geq \alpha$. Hence $\int_\alpha^a |f(x)| \geq (a-\alpha)\beta/2$ for all $a \geq \alpha$. This contradicts the fact that $f \in L^1(\mathbb{R})$. Thus, we have proved that $f(x) \to 0$ as $x \to \infty$. Similarly, we can prove that $f(x) \to 0$ as $x \to -\infty$.  

**Proof of Theorem 1.10.** We have

$$\hat{f}'(\xi) = \int_{\mathbb{R}} f'(x)e^{-ix\xi}dx.$$  

Since $f \in L^1(\mathbb{R})$,

$$\int_{\mathbb{R}} f'(x)e^{-ix\xi}dx = \lim_{a \to \infty} \int_{-a}^a f'(x)e^{-ix\xi}dx.$$  

By integration by parts, for $a > 0$,

$$\int_{-a}^a f'(x)e^{-ix\xi}dx = \left[e^{-ix\xi}f(x)\right]_{-a}^a + (i\xi) \int_{-a}^a f(x)e^{-ix\xi}dx.$$  

By Lemma 1.11, $\lim_{a \to \pm \infty} [e^{-ix\xi}f(x)]_{-a}^a = 0$. Hence,

$$\lim_{a \to \infty} \int_{-a}^a f'(x)e^{-ix\xi}dx = (i\xi) \int_{\mathbb{R}} f(x)e^{-ix\xi}dx.$$  

Thus, we have proved that $\hat{f}'(\xi) = (i\xi) \hat{f}(\xi)$.  

**Theorem 1.10** has multi-variable analogue:

For $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$ and $\alpha := (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$, define $z^\alpha := z_1^{\alpha_1}z_2^{\alpha_2} \cdots z_d^{\alpha_d}$, $|\alpha| := \alpha_1 + \cdots + \alpha_d$ and

$$\partial^\alpha \varphi := \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial x_d}\right)^{\alpha_d}.$$
For $k \in \mathbb{N}$, we write $f \in C^k(\mathbb{R}^d)$ iff $\partial^\alpha f$ exists and continuous for every $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq k$.

**Theorem 1.12.** If $f \in L^1(\mathbb{R}^d)$ such that $\partial^\alpha f$ exists a.e. and $\partial^\alpha f \in L^1(\mathbb{R}^d)$ for some multi-index $\alpha \in \mathbb{N}_0^d$, then

$$\hat{\partial^\alpha f}(\xi) = (i\xi)^\alpha \hat{f}(\xi) \quad \text{for every} \quad \xi \in \mathbb{R}^d.$$

**Theorem 1.13.** Suppose $f \in L^1(\mathbb{R})$ such that $x \mapsto g(x) := xf(x)$ belongs to $L^1(\mathbb{R})$. Then $\hat{f}$ is differentiable and

$$(\hat{f})'(\xi) = i\hat{g}(\xi), \quad \xi \in \mathbb{R}.$$

**Proof.** For $\xi, h \in \mathbb{R}$, we have

$$\hat{f}(\xi + h) - \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)[e^{-ix(\xi+h)} - e^{-ix\xi}]dx = \int_{\mathbb{R}} f(x)e^{-ix\xi}[e^{-ixh} - 1]dx$$

Hence,

$$\frac{\hat{f}(\xi + h) - \hat{f}(\xi)}{ih} = \int_{\mathbb{R}} xf(x)e^{-ix\xi}\left(e^{-ixh} - 1\right)dx$$

Note that

$$\frac{e^{-ixh} - 1}{-ix.h} \to 1 \quad \text{as} \quad |h| \to 0$$

and, since $\left|\frac{e^{-ixh} - 1}{-ix.h}\right| \leq 1$,

$$\left|\int_{\mathbb{R}} xf(x)e^{-ix\xi}\left(e^{-ixh} - 1\right)dx\right| \leq \int_{\mathbb{R}} |xf(x)|dx.$$

Hence, by DCT,

$$\frac{\hat{f}(\xi + h) - \hat{f}(\xi)}{ih} = \int_{\mathbb{R}} xf(x)e^{-ix\xi}\left(e^{-ixh} - 1\right)dx \to \int_{\mathbb{R}} (-x)f(x)e^{-ix\xi}dx$$

as $|h| \to 0$. 

More generally,

**Theorem 1.14.** If $f \in L^1(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} |x|^k|f(x)|dx < \infty$ for some $k \in \mathbb{N}$, then $\hat{f} \in C^k(\mathbb{R}^d)$ and

$$\partial^\alpha \hat{f}(\xi) = \int_{\mathbb{R}^d} (-ix)^\alpha f(x)e^{-ix\xi}dx$$

for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq k$.

Let us introduce the following operators: For $f \in L^1(\mathbb{R}^d)$,

$$(e_{ih} f)(x) = e^{ih \cdot x} f(x), \quad (\tau_h f)(x) = f(x - h), \quad (\mathcal{R} f)(x) = f(-x), \quad (D_t f)(x) = f(tx).$$
Theorem 1.15. The following results hold: For \( f \in L^1(\mathbb{R}^d), \ h \in \mathbb{R}^d, \ 0 \neq t \in \mathbb{R}, \)

1. \( \widehat{e_h f} = \tau_h \hat{f}, \)
2. \( \widehat{\tau_h f} = e_{-h} \hat{f}, \)
3. \( \mathcal{R} \hat{f} = \mathcal{R} \hat{f}, \)
4. \( \mathcal{D}_t f = |t|^{-d} D_{1/t} \hat{f}, \ 0 \neq t \in \mathbb{R}. \)

Proof. (1) For \( \xi, h \in \mathbb{R}^d, \) we have
\[
\widehat{e_h f}(\xi) = \int_{\mathbb{R}^d} e_h f(x) e^{-ix.\xi} dx = \int_{\mathbb{R}^d} e^{-ih.x} f(x) e^{-ix.\xi} dx = \int_{\mathbb{R}^d} f(x) e^{-ix(\xi-h)} dx.
\]
Thus, \( \widehat{e_h f}(\xi) = \hat{f}(\xi - h) = (\tau_h \hat{f})(\xi). \)

(2) For \( \xi, h \in \mathbb{R}^d, \) we have
\[
\widehat{\tau_h f}(\xi) = \int_{\mathbb{R}^d} \tau_h f(x) e^{-ix.\xi} dx = \int_{\mathbb{R}^d} f(x-h) e^{-ix.\xi} dx = \int_{\mathbb{R}^d} f(y) e^{-iy(\xi+h)} dx.
\]
Thus,
\[
\widehat{\tau_h f}(\xi) = \int_{\mathbb{R}^d} f(y) e^{-iy(\xi+h)} dx = e^{-ih.\xi} \int_{\mathbb{R}^d} f(y) e^{-iy.\xi} dx = e^{-ih.\xi} \hat{f}(\xi) = (e_{-h} \hat{f})(\xi).
\]

(3) For \( \xi \in \mathbb{R}^d, \) we have
\[
\mathcal{R} \hat{f}(\xi) = \int_{\mathbb{R}^d} (\mathcal{R}) f(x) e^{-ix.\xi} dx = \int_{\mathbb{R}^d} f(-x) e^{-ix.\xi} dx = \int_{\mathbb{R}^d} f(y) e^{iy.\xi} dx = \hat{f}(-\xi).
\]
Hence, \( (\mathcal{R} \hat{f})(\xi) = \hat{f}(-\xi) = (\mathcal{R} \hat{f})(\xi). \)

(4) For \( \xi \in \mathbb{R}^d \) and \( 0 \neq t \in \mathbb{R}, \) we have
\[
(\mathcal{D}_t \hat{f})(\xi) = \int_{\mathbb{R}^d} (D_t f)(x) e^{-ix.\xi} dx = \int_{\mathbb{R}^d} f(tx) e^{-ix.\xi} dx = \frac{1}{|t|^d} \int_{\mathbb{R}^d} f(y) e^{-iy.\xi/t}.
\]
Thus, \( (\mathcal{D}_t \hat{f})(\xi) = \frac{1}{|t|^d} \hat{f}(\frac{\xi}{t}) = \frac{1}{|t|^d} (D_{1/t} \hat{f})(\xi). \)

2. On surjectivity

We have seen (see Theorem 1.9) the map \( f \mapsto \hat{f} \) is a bounded linear operator from \( L^1(\mathbb{R}^d) \) to \( C_0(\mathbb{R}^d). \) A natural question is whether this operator is onto. We show that the answer is negative by using the following proposition.

Proposition 2.1. If \( f \in L^1(\mathbb{R}) \) is an odd function, then \( \hat{f} \) is an odd function and there exists \( M > 0 \) such that
\[
\left| \int_{R} \frac{\hat{f}(\xi)}{\xi} d\xi \right| \leq M
\]
for all \( r, R \) with \( 0 < r < R \).

**Proof.** Let \( f \in L^1(\mathbb{R}) \) is an odd function. It can be easily seen that\(^4\),

\[
\hat{f}(\xi) = 2i \int_0^\infty f(x) \sin(\xi x) dx.
\]

Thus, \( \hat{f} \) is odd. Let \( R \geq r > 0 \). Then, we have

\[
\int_r^R \frac{\hat{f}(\xi)}{\xi} d\xi = 2i \int_r^R \frac{1}{\xi} \left( \int_0^\infty f(x) \sin(\xi x) dx \right) d\xi
\]

\[
= 2i \int_0^\infty f(x) \left( \int_r^R \frac{\sin(\xi x)}{\xi} d\xi \right) dx
\]

\[
= 2i \int_0^\infty f(x) \left( \int_{rx}^{Rx} \frac{\sin(s)}{s} ds \right) dx.
\]

We know that there exists \( M_0 > 0 \) such that

\[
\left| \int_a^b \sin \frac{x}{x} \right| \leq M_0 \text{ for all } (a,b) \subseteq \mathbb{R}.
\]

Thus,

\[
\left| \int_r^R \frac{\hat{f}(\xi)}{\xi} d\xi \right| \leq 2 \int_0^\infty |f(x)| \left| \int_{rx}^{Rx} \frac{\sin(s)}{s} ds \right| dx \leq 2M_0 \|f\|_1.
\]

\( \square \)

**Theorem 2.2.** The bounded linear operator \( f \mapsto \hat{f} \) is a bounded linear operator from \( L^1(\mathbb{R}^d) \) to \( C_0(\mathbb{R}^d) \) is not onto.

**Proof.** By Proposition 2.1, it is enough to construct an odd function \( g \in C_0(\mathbb{R}) \) such that

\[
\left| \int_r^R \frac{g(t)}{t} dt \right| \to \infty \text{ as } R \to \infty.
\]

A candidate\(^5\) for such a function is the odd extension of \( g \) defined by

\[
g(t) := \begin{cases} 
t/e, & 0 \leq t \leq e, \\
1/\ln(t), & t > e.
\end{cases}
\]

Note that

\[
\int_e^R \frac{g(t)}{t} dt = \ln(\ln(R)) \to \infty \text{ as } R \to \infty.
\]

\( \square \)

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\(^4\)If \( f \) is a real valued even function then \( \hat{f}(\xi) = 2 \int_0^\infty f(x) \cos(\xi x) dx \).

\(^5\)This example is taken from the book *Classical Fourier Transforms*, Springer–Verlag, 1989 by K. Chandrasekharan.
3. Inversion theorem

Another question one may ask is whether $f$ can be recovered from $\hat{f}$. Here is an answer for this.

**Theorem 3.1.** (Inversion theorem) Suppose $f \in L^1(\mathbb{R})$. If $\hat{f} \in L^1(\mathbb{R})$ and if

$$g(x) := \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(t)e^{itx} dt, \quad x \in \mathbb{R},$$

then $g \in C_0(\mathbb{R})$ and $f = g$ a.e., i.e.,

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(t)e^{itx} dt, \quad a.a. \ x \in \mathbb{R}.$$  

In particular if $f \in C_0(\mathbb{R}) \cap L^1(\mathbb{R})$ such that $\hat{f} \in L^1(\mathbb{R})$, then

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(t)e^{itx} dt \quad \forall \ x \in \mathbb{R}.$$ 

An immediate corollary to the above theorem is the following.

**Corollary 3.2.** The bounded linear operator $f \mapsto \hat{f}$ from $L^1(\mathbb{R})$ into $C_0(\mathbb{R})$ is injective.

**Proof.** By the above theorem, we can infer that if $f \in L^1(\mathbb{R})$ such that $\hat{f} = 0$, then $f$ satisfies the assumption that $\hat{f} \in L^1(\mathbb{R})$ so that $f = 0$ a.e. 

Now we give two examples dealing with the cases in which the conditions of inverse theorem are satisfied/not satisfied.

**Example 3.3.** Let $f(x) = \begin{cases} 1 - |x|, & |x| \leq 1, \\ 0, & |x| > 1. \end{cases}$ Then we see that

$$\hat{f}(t) = \int_{-1}^{1} (1 - |x|)e^{-itx} dx = 2 \int_{0}^{1} (1 - x) \cos(tx)dx = \frac{\sin^2(t/2)}{(t/2)^2}.$$ 

Thus both $f$ and $\hat{f}$ belong to $L^1(\mathbb{R})$.

**Example 3.4.** Let $f(x) = \begin{cases} 1, & |x| \leq 1, \\ 0, & |x| > 1. \end{cases}$ Then we see that

$$\hat{f}(t) = \int_{-1}^{1} e^{-itx} dx = 2 \int_{0}^{1} \cos(tx)dx = 2 \left( \frac{\sin t}{t} \right).$$ 

Thus $f \in L^1(\mathbb{R})$ but $\hat{f} \notin L^1(\mathbb{R})$.
4. Proof of inversion theorem

For the proof of the inverse theorem, we shall make use of the following fact:

- If \((f_n)\) is a Cauchy sequence in \(L^1(\mathbb{R})\), then there exists an \(f \in L^1(\mathbb{R})\) and a sequence \((k_n)\) in \(\mathbb{N}\) such that \(f_{k_n} \to f\) a.e.

We are also going to make use of an integrable function \(\varphi : \mathbb{R} \to \mathbb{R}\) with the following properties:

(i) \(0 < \varphi(x) \leq 1\) for all \(x \in \mathbb{R}\);

(ii) For \(\lambda > 0\), \(\varphi_\lambda(x) := \varphi(\lambda x) \to 1\) as \(\lambda \to 0\);

(iii) The function \(\psi\) defined by

\[
\psi(x) = \frac{1}{2\pi} \int_\mathbb{R} \varphi(t) e^{itx} dt, \quad x \in \mathbb{R}
\]

is non-negative and satisfies

\[
\int_\mathbb{R} \psi(x) dx = 1.
\]

For example, one may take \(\varphi(x) = e^{-|x|}\). Clearly, this function satisfies properties (i) and (ii) above. To see (iii), we note that

\[
2\pi \psi(x) = \int_\mathbb{R} e^{-|t|} e^{itx} dt = \int_{-\infty}^{0} e^{-t} e^{itx} dt + \int_{0}^{\infty} e^{-t} e^{itx} dt = \frac{2}{1 + x^2},
\]

and hence

\[
\int_\mathbb{R} \psi(x) dx = \frac{1}{\pi} \int_\mathbb{R} \frac{dx}{1 + x^2} = 1.
\]

Thus, (iii) is also satisfied.

**Proof of inversion theorem.** Let \(\varphi : \mathbb{R} \to \mathbb{R}\) be a function satisfying the conditions (i)-(iii) above. For \(\lambda > 0\), let

\[
f_\lambda(x) := \frac{1}{2\pi} \int_\mathbb{R} \varphi_\lambda(t) \hat{f}(t) e^{itx} dt, \quad x \in \mathbb{R}.
\]

We show that

\[
\|f_\lambda - f\|_1 \to 0 \quad \text{as} \quad \lambda \to 0. \quad (1)
\]

Then\(^6\) there exists a sequence \((\lambda_n)\) of positive reals such that \(\lambda_n \to 0\) and

\[
f_{\lambda_n} \to f \quad \text{a.e.}. \quad (2)
\]

\(^6\)Recall from measure theory (cf. [1]) that every Cauchy sequence in \(L^p(\mathbb{R}^d)\) with \(1 \leq p < \infty\) has a subsequence which converges almost every where to some function in \(L^p(\mathbb{R}^d)\).
Since $0 \leq \varphi_{\lambda_n}(x) \leq 1$ and $\varphi_{\lambda_n}(x) \to 1$ for every $x \in \mathbb{R}$, we have

$$\varphi_{\lambda_n}(t)\hat{f}(t)e^{itx} \to \hat{f}(t)e^{itx}$$

and

$$|\varphi_{\lambda_n}(t)\hat{f}(t)e^{itx}| \leq |\hat{f}(t)| \quad \text{for all} \quad n \in \mathbb{N}$$

so that by DCT,

$$f_{\lambda_n}(x) := \frac{1}{2\pi} \int_{\mathbb{R}} \varphi_{\lambda_n}(t)\hat{f}(t)e^{itx}dt \to \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(t)e^{itx}dt.$$  \quad (3)

By (2) and (3),

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(t)e^{itx}dt \quad \text{for almost all} \quad x \in \mathbb{R}.$$  

Now, for proving (1), we first observe, applying Fubini’s theorem, that

$$f_{\lambda}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \varphi_{\lambda}(t)\hat{f}(t)e^{itx}dt$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \varphi(t)e^{itx} \left( \int_{\mathbb{R}} f(y)e^{-i\lambda ty}dy \right)dt$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{\lambda} \varphi(u)e^{iu(x-y)} \left( \int_{\mathbb{R}} f(y)e^{-i\lambda yu}dy \right)du$$

$$= \int_{\mathbb{R}} f(y) \left( \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{\lambda} \varphi(u)e^{i\lambda(u-y)}du \right)dy$$

$$= \int_{\mathbb{R}} f(y) \frac{1}{\lambda} \psi \left( \frac{x-y}{\lambda} \right)dy$$

$$= \int_{\mathbb{R}} f(x-\tau) \frac{1}{\lambda} \psi \left( \frac{\tau}{\lambda} \right)d\tau$$

$$= \int_{\mathbb{R}} f(x-\lambda s)\psi(s)ds.$$  

Since $\int_{\mathbb{R}} \psi(s)ds = 1$, we have

$$f_{\lambda}(x) - f(x) = \int_{\mathbb{R}} [f(x-\lambda s) - f(x)]\psi(s)ds.$$  

Hence, appealing again to Fubini,

$$\int_{\mathbb{R}} |f_{\lambda}(x) - f(x)|dx \leq \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f(x-\lambda s) - f(x)|\psi(s)ds \right)dx$$

$$\leq \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f(x-\lambda s) - f(x)|dx \right)\psi(s)ds.$$
Let
\[ g_\lambda(s) := \left( \int_{\mathbb{R}} |f(x - \lambda s) - f(x)| \, dx \right) \psi(s). \]

By Lemma 1.8, \( g_\lambda(s) \to 0 \) as \( \lambda \to 0 \). Also, \( |g_\lambda(s)| \leq 2\|f\|_1 \psi(s) \) with \( 2\|f\|_1 \psi \in L^1(\mathbb{R}) \).

Hence, by DCT,
\[ \|f_\lambda - f\|_1 := \int_{\mathbb{R}} |f_\lambda(x) - f(x)| \, dx \leq \int_{\mathbb{R}} g_\lambda(s) \, ds \to 0 \ \text{as} \ \lambda \to 0. \]

\[ \square \]

5. **The Banach algebra \( L^1(\mathbb{R}) \)**

Now, we define a multiplicative structure on \( L^1(\mathbb{R}) \)

**Definition 5.1.** For \( f, g \) in \( L^1(\mathbb{R}^d) \), we define the **convolution** of \( f \) and \( g \), denoted by \( f \ast g \), is defined by
\[ (f \ast g)(x) = \int_{\mathbb{R}^d} f(x - y)g(y) \, dy, \]
for almost all \( x \in \mathbb{R}^d \).

Note that the above definition is meaningful only if we ensure the existence of the integral involved. For this purpose we shall use the following facts.

- If \( Y \) is a topological space and \( X \) is a measurable space, then a function \( f : X \to Y \) is measurable if and only if for every Borel set \( B \) in \( Y \), \( f^{-1}(B) \) is measurable. This follows by showing that
\[ S := \{ B \subseteq Y : f^{-1}(B) \text{ measurable} \} \]
is a \( \sigma \)-algebra on \( Y \), and Borel sets belong to \( S \).

- If \( X, Y, Z \) are topological spaces and \( \varphi : X \to Y \) and \( \psi : Y \to Z \) are Borel measurable functions, then \( \psi \circ \varphi : X \to Y \) is Borel measurable.

- If \( f \) is Lebesgue measurable, then there exists a Borel measurable function \( f_0 \) such that \( f = f_0 \) a.e.

**Theorem 5.2.** Let \( f \) and \( g \) belong to \( L^1(\mathbb{R}^d) \). Then
\[ (f \ast g)(x) := \int_{\mathbb{R}^d} f(x - y)g(y) \, dy \]
is well-defined for almost all \( x \in \mathbb{R}^d \), \( f \ast g \in L^1(\mathbb{R}^d) \) and \( \|f \ast g\|_1 \leq \|f\|_1 \|g\|_1 \).
Proof. We prove the result for $d = 1$. The same arguments work for any $d \in \mathbb{N}$. First let us assume that $f$ and $g$ are Borel measurable. Note that the function $(x, y) \mapsto x - y$ is continuous from $\mathbb{R}^2$ to $\mathbb{R}$. Therefore, it can be seen that the functions $(x, y) \mapsto f(x - y)$ and $(x, y) \mapsto g(y)$ are Borel measurable whenever $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are Borel measurable. Hence, $(x, y) \mapsto f(x - y)g(y)$ is Borel measurable. Now, using the fact that $f$ and $g$ belong to $L^1(\mathbb{R})$ and Fubini’s theorem, we have

$$
\int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f(x-y)g(y)| dy \right) dx = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f(x-y)g(y)| dx \right) dy
$$

$$
= \int_{\mathbb{R}} |g(y)| \left( \int_{\mathbb{R}} |f(x-y)| dx \right) dy
$$

$$
\leq \|f\|_1 \|g\|_1.
$$

Thus, $\int_{\mathbb{R}} |f(x-y)g(y)| dy < \infty$ for all most all $x \in \mathbb{R}$, $f \ast g$ is well defined for almost all $x \in \mathbb{R}$ and $f \ast g \in L^1(\mathbb{R})$. We also have

$$
\|f \ast g\|_1 \leq \|f\|_1 \|g\|_1.
$$

Now, let us assume that $f$ and $g$ are Lebesgue measurable. Then we know that there exist Borel measurable functions $f_0$ and $g_0$ such that $f = f_0$ and $g = g_0$ a.e. Hence, we obtain the conclusion of the theorem by applying the first part to $f_0$ and $g_0$. \qed

Remark 5.3. In fact, $f \ast g$ can be defined for any $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$, where $1 \leq p < \infty$.

Theorem 5.4. The following hold.

1. $f \ast g = g \ast f$ for all $f, g \in L^1(\mathbb{R}^d)$.
2. $f \ast (g \ast h) = (f \ast g) \ast h$ for all $f, g, h \in L^1(\mathbb{R}^d)$.

Theorem 5.5. If $f, g \in L^1(\mathbb{R}^d)$, then

$$
\widehat{f \ast g} = \hat{f} \hat{g}.
$$

Proof. Note that

$$
\widehat{f \ast g} (\xi) = \int_{\mathbb{R}^d} (f \ast g)(x) e^{ix \cdot \xi} dx
$$

$$
= \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} f(x-y) e^{i(x-y) \cdot \xi} dx \right) g(y) dy
$$

$$
= \hat{f}(\xi) \hat{g}(\xi).
$$

\qed
We already know that $L^1(\mathbb{R}^d)$ is a Banach space. With respect to the multiplication $(f, g) \mapsto f \ast g$, $L^1(\mathbb{R}^d)$ a commutative Banach algebra.

**Theorem 5.6.** The Banach algebra $L^1(\mathbb{R}^d)$ does not have a multiplicative identity with respect to convolution.

**Proof.** Suppose there exists $\varphi \in L^1(\mathbb{R}^d)$ such that $f \ast \varphi = f$ for all $f \in L^1(\mathbb{R}^d)$. Then we have

$$\hat{f}(\xi)\hat{\varphi}(\xi) = \hat{f}(\xi) \quad \text{for all } \xi \in \mathbb{R}^d.$$

Taking $f(x) = e^{-|x|^2/2}$, we have $\hat{f}(\xi) = (2\pi)^{d/2}e^{-|\xi|^2/2}$, so that

$$(2\pi)^{d/2}e^{-|\xi|^2/2}\hat{\varphi}(\xi) = (2\pi)^{d/2}e^{-|\xi|^2/2} \quad \text{for all } \xi \in \mathbb{R}^d.$$

Hence, $\hat{\varphi}(\xi) = 1$ for all $\xi \in \mathbb{R}^d$. This is a contradiction to the fact that $\hat{f} \in C_0(\mathbb{R}^d)$. $\square$

However, it has an approximate identity.

**Definition 5.7.** By an approximate identity we mean a family $\{e_\lambda : \lambda > 0\}$ in $L^1(\mathbb{R}^d)$ such that $\|f \ast e_\lambda - f\|_1 \to 0$ as $\lambda \to 0$.

Now, we specify an approximate identity for $L^1(\mathbb{R})$. The same can be extended to the case of $L^1(\mathbb{R}^d)$ for appropriate changes.

Here onwards, we consider a non-negative measurable function $\psi$ which satisfies $\int_{\mathbb{R}} \psi(x)dx = 1$, and let

$$\psi_\lambda(x) := \frac{1}{\lambda} \psi\left(\frac{x}{\lambda}\right), \quad x \in \mathbb{R}, \lambda > 0.$$

Then we have

$$(f \ast \psi_\lambda)(x) = \int_{\mathbb{R}} f(x - y)\psi_\lambda(y)dy = \int_{\mathbb{R}} f(x - \lambda s)\psi(s)ds.$$

Indeed,

$$\begin{align*}
(f \ast \psi_\lambda)(x) & = \int_{\mathbb{R}} f(x - y)\psi_\lambda(y)dy \\
& = \int_{\mathbb{R}} f(y)\psi_\lambda(x - y)dy \\
& = \int_{\mathbb{R}} f(y) \frac{1}{\lambda} \psi\left(\frac{x - y}{\lambda}\right)dy \\
& = \int_{\mathbb{R}} f(x - \lambda s)\psi(s)ds.
\end{align*}$$

**Theorem 5.8.** The set $\{\psi_\lambda : \lambda > 0\}$ is an approximate identity for $L^1(\mathbb{R})$. 

Proof. Recall from (*) that
\[(f * ψ_λ)(x) = \int_{\mathbb{R}} f(x - y)ψ_λ(y)dy = \int_{\mathbb{R}} f(x - λs)ψ(s)ds. \tag{*}\]
Now, \(g_λ(s) := \int_{\mathbb{R}} |f(x - λs) - f(x)|dx ψ(s) \to 0\) as \(λ \to 0\), by Lemma 1.8, and \(g_λ(s) \leq 2\|f\|_1 ψ(s)\). Hence, by DCT,
\[\int_{\mathbb{R}} |(f * ψ_λ)(x) - f(x)|dx \to 0 \quad \text{as} \quad λ \to 0.\]
Thus, \(\|f * ψ_λ - f\|_1 \to 0\) as \(λ \to 0\). \(\square\)

In fact, something more is true.

**Lemma 5.9.** If \(f \in L^p(\mathbb{R})\) for \(1 \leq p < \infty\), then \(f * ψ_λ \in L^p(\mathbb{R})\) and
\[\lim_{λ \to 0} \|f * ψ_λ - f\|_p = 0.\]

**Proof.** Let \(f \in L^p(\mathbb{R})\) for \(1 \leq p < \infty\). We have already proved for \(p = 1\). So, assume that \(1 < p < \infty\). By (*),
\[(f * ψ_λ)(x) - f(x) = \int_{\mathbb{R}} [f(x - λs) - f(x)]ψ(s)ds.\]
Applying Hölder’s inequality with respect to the probability measure \(dμ(s) = ψ(s)ds\), we have
\[|(f * ψ_λ)(x) - f(x)| \leq \int_{\mathbb{R}} |f(x - λs) - f(x)|ψ(s)ds\]
\[\leq \left( \int_{\mathbb{R}} |f(x - λs) - f(x)|^p ψ(s)ds \right)^{1/p} \left( \int_{\mathbb{R}} ψ(s)ds \right)^{1/q}\]
\[\leq \left( \int_{\mathbb{R}} |f(x - λs) - f(x)|^p ψ(s)ds \right)^{1/p}.\]
Hence, applying Fubini’s theorem,
\[\int_{\mathbb{R}} |(f * ψ_λ)(x) - f(x)|^p dx \leq \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f(x - λs) - f(x)|^p ψ(s)ds \right)dx\]
\[= \int_{\mathbb{R}} ψ(s) \left( \int_{\mathbb{R}} |f(x - λs) - f(x)|^p dx \right) ds.\]
Since \(\int_{\mathbb{R}} ψ(s)ds = 1\), we have
\[\int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f(x - λs) - f(x)|^p dx \right) ψ(s)ds \leq (2\|f\|_p)^p\]
so that $f \ast \psi_\lambda \in L^p$. Further, $\psi(s) \int_\mathbb{R} |f(x - \lambda s) - f(x)|^p dx \leq (2\|f\|_p^p)\psi(s)$ and by Lemma 1.8,

$$\psi(s) \int_\mathbb{R} |f(x - \lambda s) - f(x)|^p dx \to 0 \text{ as } \lambda \to 0.$$ 

Hence, by DCT,

$$\|f \ast \psi_\lambda - f\|_p = \int_\mathbb{R} |(f \ast \psi_\lambda)(x) - f(x)|^p dx \leq \int_\mathbb{R} \int_\mathbb{R} \psi(s)|f(x - \lambda s) - f(x)|^p dx ds \to 0$$

as $\lambda \to 0$. \hfill $\Box$

We observe that for each $t \in \mathbb{R}$, the linear functional $\varphi_t : L^1(\mathbb{R}) \to \mathbb{R}$ defined by $\varphi_t(f) = \hat{f}(t)$, $f \in L^1(\mathbb{R})$ is a linear functional which also satisfies the multiplicative property:

$$\varphi_t(f \ast g) = \varphi_t(f)\varphi_t(g), \quad f,g \in L^1(\mathbb{R}).$$

In other words, $\varphi_t$ is a complex homomorphism on Banach algebra $L^1(\mathbb{R})$. This linear functional is continuous: For each $t \in \mathbb{R}$,

$$|\varphi_t(f)| = |\hat{f}(t)| \leq \|f\|_1 \quad \text{for all } f \in L^1(\mathbb{R}).$$

It also shows that $\|\varphi_t\| \leq 1$ for all $t \in \mathbb{R}$. In fact, this is true for every complex homomorphism on any Banach algebra as the following theorem shows.

**Theorem 5.10.** Let $\mathcal{A}$ be a complex Banach algebra and let $\varphi : \mathcal{A} \to \mathbb{C}$ be a algebra homomorphism. Then $\varphi$ is continuous and $\|\varphi\| \leq 1$.

**Proof.** We prove that $\|f(a)\| \leq \|a\|$ for every $a \in \mathcal{A}$: Suppose this is not true. then there exists $a_0 \in \mathcal{A}$ such that $|\varphi(a_0)| > \|a_0\|$. Let $a = a_0/|\varphi(a_0)|$. Then $\|a\| < 1$ and $\varphi(a) = 1$. Since $a^n \to 0$, the sequence $(b_n)$ defined by $b_n = -(a + a^2 + \cdots + a^n)$ is convergent, say $b_n \to b$. Note that

$$ab_n = -(a^2 + a^3 + \cdots + a^{n+1}) = a + b_n - a^{n+1}.$$ 

Taking limits, $ab = a + b$. Hence,

$$\varphi(a) + \varphi(b) = \varphi(ab) = \varphi(a)\varphi(b).$$

Since $\varphi(a) = 1$, we obtain $1 + \varphi(b) = \varphi(b)$. This is impossible. \hfill $\Box$

The following theorem shows that Fourier transform is the only complex linear functional on $L^1(\mathbb{R})$ having the multiplicative property.

**Theorem 5.11.** If $\varphi : L^1(\mathbb{R}) \to \mathbb{C}$ is a multiplicative linear functional, that is, $\varphi$ is a linear functional satisfying

$$\varphi(f \ast g) = \varphi(f)\varphi(g) \quad \forall f,g \in L^1(\mathbb{R}),$$

then there exists $t \in \mathbb{R}$ such that $\varphi(f) = \hat{f}(t)$ for all $f \in L^1(\mathbb{R})$. 
Proof. Let $\varphi$ be a nonzero continuous linear functional on $L^1(\mathbb{R})$. Then by Riesz representation theorem, there exists a unique $h \in L^\infty(\mathbb{R})$ such that

$$\varphi(f) = \int_{\mathbb{R}} f(x)h(x)dx \quad \text{for every} \quad f \in L^1(\mathbb{R}).$$

Using the fact that $\varphi(f \ast g) = \varphi(f)\varphi(g)$, we obtain the relation

$$\int_{\mathbb{R}} g(y)\varphi(f_y)dy = \varphi(f) \int_{\mathbb{R}} g(y)h(y)dy$$

where $f_y(x) := f(x-y)$. From this, $\varphi(f_y) = \varphi(f)h(y)$. Since $y \mapsto f_y$ is continuous, we may assume, without loss of generality, that $h$ is continuous. We also have

$$\varphi(f)h(x+y) = \varphi(f)h(x) = \varphi(f_x)(y) = \varphi(f_x)h(y) = \varphi(f)(x)h(y).$$

Hence, $h(x+y) = h(x)h(y)$ and hence, $h(0) = 1$, as $h \neq 0$. Continuity of $h$ implies the existence of $\delta > 0$ such that $\int_0^\delta h(y)dy \neq 0$. Now, from $h(x+y) = h(x)h(y)$ we have

$$h(x) \int_0^\delta h(y)dy = \int_0^\delta h(x+y)dy = \int_x^{x+\delta} h(t)dt.$$  

Hence, $h$ is differentiable so that from $h(x+y) = h(x)h(y)$, $h'(x) = h(x)h'(0)$, i.e.,

$$h'(x) = \alpha h(x), \quad \alpha := h'(0).$$

Thus,

$$h(x) = e^{\alpha x}.$$

Since $h$ is bounded, $\alpha := -i\tau$ for some $\tau \in \mathbb{R}$. Thus, $h(x) = e^{-i\tau x}$ for some $\tau \in \mathbb{R}$. \qed

Remark 5.12. The above proof is adopted from Rudin[3].

6. Fourier-Plancheral Transform on $L^2(\mathbb{R})$

We have already observed that $f \mapsto \hat{f}$ is an injective bounded linear operator from $L^1(\mathbb{R})$ into $C_0(\mathbb{R})$. We have also given an example to show that this map is not surjective. Next theorem introduce the definition of Fourier transform of functions in $L^2(\mathbb{R})$ by extending the known definition from $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ to all of $L^2(\mathbb{R})$.

Lemma 6.1. Let $f \in C_b(\mathbb{R})$. Then

$$\lim_{\lambda \to 0}(f \ast \psi_\lambda)(x) = f(x) \quad \text{for every} \quad x \in \mathbb{R},$$

where $\psi_\lambda$ is as in Theorem 5.8.
Proof. Note that for every $x \in \mathbb{R}$,

$$(f * \psi_f)(x) - f(x) = \int_{\mathbb{R}} [f(x - \lambda s) - f(x)] \psi(s) ds.$$ 

Note that $|f(x - \lambda s) - f(x)| \psi(s) \leq 2\|f\|_\infty \psi(s)$ with $\psi \in L^1(\mathbb{R})$ and

$$[f(x - \lambda s) - f(x)] \psi(s) \to 0 \quad \text{as} \quad \lambda \to 0$$

as $f$ is continuous at $x$. Hence, the result follows by DCT. 

\[\square\]

Theorem 6.2. The following hold.

(i) If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then $\hat{f} \in L^2(\mathbb{R})$ and $\|\hat{f}\|_2 = \sqrt{2\pi} \|f\|_2$.

(ii) The set $Y := \{\hat{f} : f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})\}$ is dense in $L^2(\mathbb{R})$.

Proof. (i) Let $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. To show that $|\hat{f}|^2 \in L^1(\mathbb{R})$. Note that

$|\hat{f}|^2 = \tilde{f} \bar{\tilde{f}},$

$$\tilde{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi} dx = \int_{\mathbb{R}} \tilde{f}(x) e^{ix\xi} dx = \int_{\mathbb{R}} \tilde{f}(-x) e^{-ix\xi} dx = \int_{\mathbb{R}} \tilde{f}(x) e^{-it\xi} dt,$$

where $\tilde{f}(x) = \overline{f(-x)}$ for $x \in \mathbb{R}$. Thus, $\tilde{f}(\xi) = \bar{\tilde{f}}(\xi)$ and hence,

$$|\hat{f}|^2 = \tilde{f} \bar{\tilde{f}} = \tilde{f} \hat{f} = f * \tilde{f} = g,$$

where $g := f * \tilde{f}$. Note that

$$g(x) = \int_{\mathbb{R}} f(x - y) \tilde{f}(y) dy = \int_{\mathbb{R}} f(x - y) \overline{f(-y)} dy = \int_{\mathbb{R}} f(x + u) \overline{f(u)} du = \langle f_{-x}, f \rangle,$$

where $f_{\tau}(x) := f(x - \tau)$, and

$$|g(x)| = |\langle f_{-x}, f \rangle| \leq \|f\|_2^2.$$

Thus, $g$ is bounded. By Lemma 1.8, $\tau \mapsto f_{\tau}$ from $\mathbb{R}$ to $L^2(\mathbb{R})$ is continuous so that $g$ is continuous. Hence, by Lemma 6.1,

$$(g * \psi_f)(x) \to g(x) \quad \text{as} \quad \lambda \to 0^+$$

for every $x \in \mathbb{R}$. In particular,

$$(g * \psi_f)(0) \to g(0) = \|f\|_2^2 \quad \text{as} \quad \lambda \to 0^+.$$
But, taking \( \varphi \) is as in the proof of inversion theorem with corresponding \( \psi \),

\[
(g \ast \psi_\lambda)(x) = \int_\mathbb{R} g(y) \frac{1}{\lambda} \psi \left( \frac{x - y}{\lambda} \right) dy
\]

\[
= \int_\mathbb{R} g(y) \left( \frac{1}{2\pi} \int_{\mathbb{R}} \varphi(u) e^{iux/y} du \right) dy
\]

\[
= \frac{1}{2\pi} \int_{\mathbb{R}} \varphi(u) \left( \int_{\mathbb{R}} g(y) e^{-iuy/\lambda} dy \right) du
\]

\[
= \frac{1}{2\pi} \int_{\mathbb{R}} \varphi(\lambda t) e^{itx} \left( \int_{\mathbb{R}} g(y) e^{-iyt} dy \right) dt
\]

\[
= \frac{1}{2\pi} \int_{\mathbb{R}} \varphi(\lambda t) \hat{g}(t) e^{itx} dt.
\]

In particular,

\[
(g \ast \psi_\lambda)(0) = \frac{1}{2\pi} \int_{\mathbb{R}} \varphi(\lambda t) \hat{g}(t) dt.
\]

Recall that \( \hat{g}(t) = |\hat{f}|^2 \geq 0 \). Hence, assuming that \( \varphi(\lambda t) \) increases to 1 as \( \lambda \to 0^+ \) (e.g., \( \varphi(x) = e^{-|x|} \)), we obtain \( \varphi(\lambda t) \hat{g}(t) \) increases to \( \hat{g}(t) \) as \( \lambda \to 0^+ \). Hence, by MCT

\[
(g \ast \psi_\lambda)(0) = \frac{1}{2\pi} \int_{\mathbb{R}} \varphi(\lambda t) \hat{g}(t) dt \to \frac{1}{2\pi} \int_{\mathbb{R}} \hat{g}(t) dt = \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{f}(t)|^2 dt \quad \text{as} \quad \lambda \to 0^+,
\]

We already have \( (g \ast \psi_\lambda)(0) \to ||f||^2 \) as \( \lambda \to 0^+ \). Thus,

\[
\frac{1}{2\pi} \int_{\mathbb{R}} |\hat{f}(t)|^2 dt = ||f||^2
\]

so that \( \hat{f} \in L^2(\mathbb{R}) \) and \( ||\hat{f}||^2 = 2\pi ||f||^2 \).

(ii) It is enough to show that \( Y^\perp = \{0\} \), i.e., to prove that \( w \in Y^\perp \) implies \( w = 0 \). So, let \( w \in Y \). By Lemma 5.9, \( \psi_\lambda \ast w \in L^2(\mathbb{R}) \) and \( \|\psi_\lambda \ast w - w\|_2 \to 0 \) as \( \lambda \to 0^+ \). Hence, it is enough to prove that \( \psi_\lambda \ast w = 0 \) for all \( \lambda > 0 \).

Note that

\[
(\psi_\lambda \ast w)(x) := \int_{\mathbb{R}} \psi_\lambda(x - y) w(y) dy, \quad x \in \mathbb{R}.
\]

Taking \( \varphi \) and \( \psi \) as in the proof of inversion theorem, we have

\[
\psi_\lambda(x - y) = \frac{1}{2\pi} \int_{\mathbb{R}} \varphi(\lambda t) e^{it(x - y)} dt = \frac{1}{2\pi} \int_{\mathbb{R}} \varphi(\lambda t) e^{itx} e^{-ity} dt.
\]

Assuming \( \varphi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \) (e.g. \( \varphi(t) = e^{-|t|} \)), by the above relation, we have \( y \mapsto \psi_\lambda(x - y) \) is a Fourier transform of a function in \( L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \) so that it belongs to \( Y \). Thus, \( \psi_\lambda \ast w = 0 \) for all \( \lambda > 0 \). This completes the proof. \( \square \)

In view of the above theorem, we have the following.
Theorem 6.3. (Fourier-Plancheral Theorem) There exists a unique surjective continuous linear operator $\Phi$ from $L^2(\mathbb{R})$ onto itself such that

$$\Phi(f) = \hat{f} \quad \text{for every} \quad f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$$

and $\|\Phi(f)\|_2 = \sqrt{2\pi} \|f\|_2$ for every $f \in L^2(\mathbb{R})$.

Proof. By Theorem 6.2, the map Define $\Phi_0 : L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined by $\Phi_0(f) = \hat{f}$, $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, is a continuous linear operator with its domain and range dense in $L^2(\mathbb{R})$. Recall also that $\|\Phi_0(f)\|_2 = \|\hat{f}\|_2 = \sqrt{2\pi} \|f\|_2$. Hence, it has a unique continuous linear extension $\Phi : L^2(\mathbb{R}) \to L^2(\mathbb{R})$. Also, for $f \in L^2(\mathbb{R})$, if $(f_n)$ in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is such that $\|f - f_n\|_2 \to 0$, then we have

$$\|\Phi(f)\|_2 = \lim_{n \to \infty} \|\Phi(f_n)\|_2 = \lim_{n \to \infty} \|\hat{f}_n\|_2 = \lim_{n \to \infty} \sqrt{2\pi} \|f_n\|_2 = \sqrt{2\pi} \|f\|_2.$$ 

This also shows that range of $\Phi$ is closed. Since range of $\Phi_0$ is dense in $L^2(\mathbb{R})$, $\Phi$ is onto. \[\square\]

Definition 6.4. The operator $\Phi : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ in Theorem 6.3 is called the Fourier-Plancheral transform.

Notation 6.5. Abusing the notation, for $f \in L^2(\mathbb{R})$, $\Phi(f)$ is also denoted by $\hat{f}$. This is somewhat justified by Theorem 6.9 below.

Now, let us observe the following.

Theorem 6.6. Let $f \in L^1(\mathbb{R}^d)$ and for $n > 0$, let $E_n := \{x \in \mathbb{R}^d : |x| \leq n\}$. Then

$$\hat{f}(\xi) = \lim_{n \to \infty} \int_{E_n} f(x)e^{-ix.\xi}dx$$

for every $\xi \in \mathbb{R}^d$.

Proof. Let $\xi \in \mathbb{R}^d$. Since $\bigcup_{n=1}^\infty E_n = \mathbb{R}^d$ and $E_n \subseteq E_{n+1}$ for every $n \in \mathbb{N}$, we have $(\chi_{E_n}f)(x)e^{-ix.\xi} \to f(x)e^{-ix.\xi}$ as $n \to \infty$. Also,

$$|(\chi_{E_n}f)(x)e^{-ix.\xi}| \leq |f(x)|, \quad f \in L^1(\mathbb{R}^d).$$

Hence, by DCT,

$$\int_{E_n} f(x)e^{-ix.\xi}dx \to \int_{\mathbb{R}^d} f(x)e^{-ix.\xi}dx = \hat{f}(\xi) \quad \text{as} \quad n \to \infty$$

for every $\xi \in \mathbb{R}^d$. \[\square\]

What about for $f \in L^2(\mathbb{R}^d)$? Here is the answer.
Theorem 6.7. Let \( f \in L^2(\mathbb{R}) \) and for \( n > 0 \), let \( E_n := \{ x \in \mathbb{R}^d : |x| \leq n \} \). The following hold.

1. \( f_n := \chi_{E_n} f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \) for every \( n \in \mathbb{N} \).
2. \( \| f - f_n \|_2 \to 0 \) as \( n \to \infty \).
3. \( (f_n) \) is a Cauchy sequence in \( L^2(\mathbb{R}) \).
4. \( \| \hat{f}_n - \Phi(f) \|_2 \to 0 \) as \( n \to \infty \), where \( \Phi \) is the Fourier-Plancheral transform.
5. There exists a subsequence \( (\hat{f}_{k_n}) \) for \( (\hat{f}_n) \) such that \( \hat{f}_{k_n} \to \Phi(f) \) a.e. on \( \mathbb{R} \).

Proof. (1) We have
\[
\int_{\mathbb{R}} |f_n(x)| dx = \int_{\mathbb{R}} \chi_{E_n}(x)|f(x)| dx \leq \|f\|_2 [\mu(E_n)]^{1/2},
\]
\[
\int_{\mathbb{R}} |f_n(x)|^2 dx \leq \|f\|_2^2.
\]
Hence, \( f_n := \chi_{E_n} f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \) for every \( n \in \mathbb{N} \).

(2) We have
\[
\int_{\mathbb{R}} |f - f_n(x)|^2 dx = \int_{\mathbb{R}} [1 - \chi_{E_n}(x)] |f(x)|^2 dx = \int_{\mathbb{R}} \chi_{E_n^c}(x) |f(x)|^2 dx,
\]
\( \chi_{E_n^c}(x) |f(x)|^2 \to 0 \) as \( n \to \infty \) and \( \chi_{E_n}(x) |f(x)|^2 \leq |f(x)|^2 \) with \( |f|^2 \in L^1(\mathbb{R}) \) so that by DCT, \( \|f - f_n\|_2^2 = \int_{\mathbb{R}} |f - f_n(x)|^2 dx \to 0 \) as \( n \to \infty \).

(3) We have
\[
\|\hat{f}_n - \hat{f}_m\|_2 = \|f_n - f_m\|_2 = \sqrt{2\pi} \|f_n - f_m\|_2.
\]
Hence, by (2), \( \|\hat{f}_n - \hat{f}_m\|_2 = \sqrt{2\pi} \|f_n - f_m\|_2 \to 0 \) as \( n, m \to \infty \).

(4) By (2) and Fourier-Plancheral theorem, we have
\[
\|\hat{f}_n - \Phi(f)\|_2 = \|\Phi(f_n) - \Phi(f)\|_2 = \sqrt{2\pi} \|f_n - f\|_2 \to 0 \quad \text{as} \quad n \to \infty.
\]

(5) This follows from (4). \( \square \)

Remark 6.8. For \( f \in L^2(\mathbb{R}) \), let
\[
P_rf = f\chi_{[-r,r]}, \quad r > 0.
\]
Then \( P_r f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \) and \( P_r : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) is an orthogonal projection satisfying
\[
\|P_rf - f\|_2 \to 0 \quad \text{and} \quad \|P_rf - \Phi(f)\|_2 \to 0 \quad \text{as} \quad r \to \infty.
\]
**Theorem 6.9.** For \( f \in L^2(\mathbb{R}) \) and \( r > 0 \), let
\[
g_r(t) := \int_{-r}^{r} f(x)e^{-itx}dx, \quad h_r(x) := \int_{-r}^{r} \Phi(f)(t)e^{itx}dt
\]
for all \( x, t \in \mathbb{R} \). Then \( g_r \in L^2(\mathbb{R}) \), \( h_r \in L^2(\mathbb{R}) \), and
\[
\|g_r - \Phi(f)\|_2 \to 0, \quad \|h_r - f\|_2 \to 0 \quad \text{as} \quad r \to \infty.
\]

**Proof.** Exercise. \( \square \)

### 7. Problems

(1) Let \( f \in L^1(\mathbb{R}^d) \). Prove the following:
   (a) \( \hat{f}(\xi) \) is well-defined for every \( \xi \in \mathbb{R}^d \).
   (b) \( \sup_{\xi \in \mathbb{R}^d} |\hat{f}(\xi)| \leq \|f\|_1 \).
   (c) \( \xi \to \hat{f}(\xi) \) is uniformly continuous on \( \mathbb{R}^d \).
   (d) The map \( f \mapsto \hat{f} \) is a linear operator from \( L^1(\mathbb{R}^d) \) to \( C_b(\mathbb{R}^d) \) with norm at most 1.
   (e) \( |\hat{f}(\xi)| \to 0 \) as \( |\xi| \to 0 \).

(2) Let \( 1 \leq p < \infty \). For \( f \in L^p(\mathbb{R}^d) \) and \( y \in \mathbb{R}^d \), let
\[
(\tau_y f)(x) := f(x - y), \quad x \in \mathbb{R}^d.
\]
For each \( f \in L^p(\mathbb{R}^d) \), prove the following:
   (a) \( \tau_y f \in L^p(\mathbb{R}^d) \) for every \( y \in \mathbb{R}^d \).
   (b) The map \( y \mapsto \tau_y f \) from \( \mathbb{R}^d \) to \( L^p(\mathbb{R}^d) \) is continuous.

(3) Prove that, if \( f \in L^1(\mathbb{R}) \) is differentiable with \( f' \in L^1(\mathbb{R}) \), then
\[
\hat{f}'(\xi) = (i\xi)\hat{f}(\xi), \quad \xi \in \mathbb{R}.
\]

(4) Suppose \( f \in L^1(\mathbb{R}) \) such that \( x \mapsto g(x) := xf(x) \) belongs to \( L^1(\mathbb{R}) \). Prove that \( \hat{f} \) is differentiable and
\[
(\hat{f})'(\xi) = i\hat{g}(\xi), \quad \xi \in \mathbb{R}.
\]

(5) Let
\[
(e_h f)(x) = e^{ihx} f(x), \quad (\tau_h f)(x) = f(x - h), \quad (\mathcal{R} f)(x) = f(-x), \quad (D_t f)(x) = f(tx).
\]
For \( f \in L^1(\mathbb{R}^d) \), \( h \in \mathbb{R}^d \), \( 0 \neq t \in \mathbb{R} \), prove the following.
   (a) \( e_h \hat{f} = \tau_h \hat{f} \).
   (b) \( \tau_h \hat{f} = e_{-h} \hat{f} \).
   (c) \( \mathcal{R} \hat{f} = \mathcal{R} \hat{f} \).
   (d) \( D_t \hat{f} = |t|^{-d} D_{1/t} \hat{f}, \quad 0 \neq t \in \mathbb{R} \).
(6) Prove that the operator $f \mapsto \hat{f}$ from $L^1(\mathbb{R}^d)$ to $C_0(\mathbb{R}^d)$ is not onto.
(7) Suppose $f \in L^1(\mathbb{R})$. If $\hat{f} \in L^1(\mathbb{R})$ and if
\[ g(x) := \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(t)e^{itx} dt, \quad x \in \mathbb{R}, \]
then prove that $f = g$ a.e. Also deduce the following.
(a) If $f \in C_0(\mathbb{R}) \cap L^1(\mathbb{R})$ such that $\hat{f} \in L^1(\mathbb{R})$, then
\[ f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(t)e^{itx} dt \quad \forall x \in \mathbb{R}. \]
(b) $f \mapsto \hat{f}$ is an injective operator.
(8) Let $\varphi : \mathbb{R} \to \mathbb{R}$ be an integrable function such that $0 < \varphi(x) \leq 1$ for all $x \in \mathbb{R}$ and $\varphi(\lambda x) := \varphi(\lambda x) \to 1$ as $\lambda \to 0^+$. Prove that, for every $g \in L^1(\mathbb{R})$,
\[ \int_{\mathbb{R}} \varphi(\lambda t)g(t)e^{itx} dt \to \int_{\mathbb{R}} g(t)e^{itx} dt. \]
(9) Let $\varphi$ be as in Problem (8) and let $\psi$ be defined by $\psi(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \varphi(t)e^{itx} dt$ for $x \in \mathbb{R}$. If $f \in L^1(\mathbb{R})$ is such that $\hat{f} \in L^1(\mathbb{R})$ and if $\int_{\mathbb{R}} \psi(x)dx = 1$, then prove that
\[ \frac{1}{2\pi} \int_{\mathbb{R}} \varphi(\lambda t)\hat{f}(t)e^{itx} dt = \int_{\mathbb{R}} f(x - \lambda s)\psi(s)ds. \]
(10) Let $\psi \in L^1(\mathbb{R})$ be a non-negative function such that $\int_{\mathbb{R}} \psi(x)dx = 1$ and for $f \in L^1(\mathbb{R})$ and $\lambda > 0$, let
\[ h_\lambda(x) := \int_{\mathbb{R}} f(x - \lambda s)\psi(s)ds. \]
Prove that $\|h_\lambda - f\|_1 \to 0$ as $\lambda \to 0$. Deduce the following.
(a) There exists a sequence $(\lambda_n)$ of positive real numbers such that $h_{\lambda_n} \to f$ a.e.
(b) If $\varphi$ is as in Problem (8), $f \in L^1(\mathbb{R})$ such that $\hat{f} \in L^1(\mathbb{R})$, and
\[ f_\lambda(x) := \int_{\mathbb{R}} \varphi(\lambda t)\hat{f}(t)e^{itx} dt, \]
then $\|f_\lambda - f\|_1 \to 0$ as $\lambda \to 0$.
(11) Let $\varphi(x) = e^{-|x|}$. Verify the following.
(a) $\varphi$ satisfies the conditions in Problem (8).
(b) $\psi(x) := \frac{1}{2\pi} \int_{\mathbb{R}} \varphi(t)e^{itx} dt$, $x \in \mathbb{R}$ satisfies $\int_{\mathbb{R}} \psi(x)dx = 1$.
(12) Let $\psi$ be as in Problem (9). If $f \in L^p(\mathbb{R})$ for $1 \leq p < \infty$, then prove that $f * \psi_\lambda \in L^p(\mathbb{R})$ and $\lim_{\lambda \to 0} \|f * \psi_\lambda - f\|_p = 0$.
(13) Let $\psi$ be as in Problem (9) and $f \in L^\infty(\mathbb{R})$. Prove that if $f$ is continuous at a point $x \in \mathbb{R}$, then $\lim_{\lambda \to 0}(f * \psi_\lambda)(x) = f(x)$, where $\psi_\lambda$ is as in Problem (9).
(14) Prove the following.
(a) If \( f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \), then \( \hat{f} \in L^2(\mathbb{R}) \) and \( \|\hat{f}\|_2 = 2\pi\|f\|_2 \).
(b) The set \( Y := \{ \hat{f} : f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \} \) is dense in \( L^2(\mathbb{R}) \).

(15) Prove that there exists a unique surjective continuous linear operator \( \Phi \) from \( L^2(\mathbb{R}) \) onto itself such that \( \Phi(f) = \hat{f} \) for every \( f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \). Show also that \( \|\Phi(f)\|_2 = 2\pi\|f\|_2 \) for every \( f \in L^2(\mathbb{R}) \).

(16) Let \( f \in L^1(\mathbb{R}^d) \) and for \( n > 0 \), let \( E_n := \{ x \in \mathbb{R}^d : |x| \leq n \} \). Prove that
\[
\int_{E_n} f(x) e^{-ix.\xi} dx \to \int_{\mathbb{R}^d} f(x) e^{-ix.\xi} dx = \hat{f}(\xi) \quad \text{as} \quad n \to \infty
\]
for every \( \xi \in \mathbb{R}^d \).

(17) Let \( f \in L^2(\mathbb{R}) \) and for \( n > 0 \), let \( E_n := \{ x \in \mathbb{R}^d : |x| \leq n \} \). Prove the following.
(a) \( f_n := \chi_{E_n} f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \) for every \( n \in \mathbb{N} \).
(b) \( \|f - f_n\|_2 \to 0 \) as \( n \to \infty \).
(c) \( (\hat{f}_n) \) is a Cauchy sequence in \( L^2(\mathbb{R}) \).
(d) \( \|\hat{f}_n - \Phi(f)\|_2 \to 0 \) as \( n \to \infty \), where \( \Phi \) is the Fourier-Plancheral transform.
(e) There exists a subsequence \( (\hat{f}_{k_n}) \) for \( (\hat{f}_n) \) such that \( \hat{f}_{k_n} \to \Phi(f) \) a.e. on \( \mathbb{R} \).

(18) For \( f \in L^2(\mathbb{R}) \) and \( r > 0 \), let
\[
g_r(\xi) := \int_{-r}^{r} f(x) e^{-ix.\xi} dx, \quad h_r(x) := \int_{-r}^{r} \Phi(f)(\xi)e^{ix.\xi} d\xi
\]
for all \( x, \xi \in \mathbb{R} \). Then \( g_r \in L^2(\mathbb{R}) \), \( h_r \in L^2(\mathbb{R}) \), and
\[
\|g_r - \Phi(f)\|_2 \to 0, \quad \|h_r - f\|_2 \to 0 \quad \text{as} \quad r \to \infty.
\]

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