Measure and Integration: A Crash Course*

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Abstract

In these notes we intend to give a crash course on measure and integration motivating the essential concepts of

1. Lebesgue measure on $\mathbb{R}$,
2. Lebesgue measurable sets ,
3. σ-algebra $\mathcal{M}$ of measurable subsets of $\mathbb{R}$,
4. Lebesgue measure on measurable subsets of $\mathbb{R}$,
5. measure μ on a general σ-algebra $\mathcal{A}$ on a none-empty set Ω,
6. measurable function on a general measurable space $(X, \mathcal{A})$,
7. integral of positive, real and complex valued measurable functions on a measure space $(X, \mathcal{A}, \mu)$,
8. L^p-spaces and Sobolev spaces.

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1 Lecture 1: Lebesgue measure

1.1 The need for the concept of a measure

We recall the following:

- Every continuous function $f : [a, b] \to \mathbb{R}$ is Riemann integrable.
- Every bounded function $f : [a, b] \to \mathbb{R}$ which is discontinuous only at a finite number of points in $[a, b]$ is Riemann integrable.
- Every function $f : [a, b] \to \mathbb{R}$ which is either monotonically increasing or monotonically decreasing is Riemann integrable.

Further, it is also known that

- Every bounded function $f : [a, b] \to \mathbb{R}$ which is discontinuous only on a *set of measure zero* is Riemann integrable.

In the last statement, by a *set of measure zero* we mean a set $E \subseteq \mathbb{R}$ such that for every $\varepsilon > 0$, there exists a countable collection $\{I_n\}$ of open intervals such that

$$E \subseteq \bigcup_n I_n \quad \text{and} \quad \sum_n \ell(I_n) < \varepsilon.$$

However, the concept of Riemann integral has certain draw backs:

- If (f_n) is a sequence of Riemann integrable functions on $[a, b]$ such that $f_n(x) \to f(x)$ for every $x \in [a, b]$ for some bounded function f, then f need not be Riemann integrable.

Exercise 1.1. Let

$$f_n(x) = \begin{cases} 1 & \text{if } x \in \{r_1, \ldots, r_n\} \\ 0 & \text{if } x \notin \{r_1, \ldots, r_n\}, \end{cases}$$

Show that each f_n is Riemann integrable, and $(f_n(x))$ converges for every $x \in [0, 1]$, but the limit function is not Riemann integrable. ♢

We would like to extend the concept of integrability to a larger class of functions so that the draw back of Riemann integral as in Example 1.1, does not occur. This is done by using the concept of a *measure* of a subset of $\mathbb{R}$ which is a generalization of the concept of *length* of an interval.

1.2 Lebesgue outer measure on $\mathbb{R}$

Definition 1.2. For $E \subseteq \mathbb{R}$, the **Lebesgue outer measure** of E is defined as

$$m^*(E) := \inf_{\mathcal{I}} \sum_n \ell(I_n),$$

where the infimum is taken over the set $\mathcal{I}$ of all countable collection $\{I_n\}$ of open intervals such that $E \subseteq \cup_n I_n$. ♢

Note that $0 \leq m^*(E) \leq \infty$ for every $E \subseteq \mathbb{R}$. Thus, m^* can be thought of as a function from the family of all subsets of $\mathbb{R}$ into the set $[0, \infty]$.

Exercise 1.3. Let $E \subseteq \mathbb{R}$. Prove that for every $\varepsilon > 0$ there exists a countable family $\{I_n\}$ of open intervals such that $E \subseteq \bigcup_n^{\infty} I_n$ and $\sum_n \ell(I_n) \leq m^*(E) + \varepsilon$. Strict inequality occurs in the above if $m^*(E) < \infty$. ♢

Exercise 1.4. Prove the following:

(i) $m^*(\varnothing) = 0$.

(ii) If E is a singleton set, then $m^*(E) = 0$.

(iii) If E is a countable set, then $m^*(E) = 0$.

(iv) If $E = (a, b)$ with $a < b$ then $m^*(E) \leq b - a$. ♢

Exercise 1.5. Prove the following:

(i) $E_1 \subseteq E_2 \Rightarrow m^*(E_1) \leq m^*(E_2)$.

(ii) $m^*(A \cup B) \leq m^*(A) + m^*(B)$.

(iii) If $E \subseteq A$ and $m^*(E) = 0$, then $m^*(A \setminus E) = m^*(A)$.

(iv) If $E \subseteq \mathbb{R}$ and $x \in \mathbb{R}$, then $m^*(E + x) = m^*(E)$. ♢

Remark 1.6. The property (i) in Exercise 1.5 is called the *monotonicity* property of m^*, and the property (iv) is called the *translation invariance* of m^*. ♢

Exercise 1.7. Prove that for every sequence (I_n) of open intervals,
$m^*(\bigcup I_n) \leq \sum_n \ell(I_n)$. ♢

Exercise 1.8. If $E \subseteq \mathbb{R}$, then for every $\varepsilon > 0$, there exists an open set $G \subseteq \mathbb{R}$ such that $G \supseteq E$ and $m^*(G) \leq m^*(E) + \varepsilon$. ♢

THEOREM 1.9. (Sub-additivity) *Suppose $A_n \subseteq \mathbb{R}$ for $n \in \mathbb{N}$. Then*

$$m^* \left(\bigcup_{n=1}^{\infty} A_n \right) \leq \sum_{n=1}^{\infty} m^*(A_n).$$

Proof. Use Exercise 1.3. □

Exercise 1.10. Derive Exercise 1.4(iii) from Exercise 1.4(ii) by using Theorem 1.9 ♢

THEOREM 1.11. *The Lebesgue outer measure of any interval is its length.*

Proof. Let I be an interval.
Case 1: Suppose $I = [a, b]$ with $-\infty < a < b < \infty$, and let $\varepsilon > 0$. Let $I_\varepsilon := (a - \varepsilon, b + \varepsilon)$. Clearly,

$$m^*(I) \leq m^*(I_\varepsilon) \leq b - a + 2\varepsilon.$$

This is true for all $\varepsilon > 0$. Hence, $m^*(I) \leq b - a$.

Also, there exists a countable collection $\{I_n\}$ of open intervals of finite length such that $I \subseteq \cup_n I_n$ and $\sum_n \ell(I_n) \leq m^*(I) + \varepsilon$. By compactness of I, there exists a finite sub-collection $\{I_{n_1},, \ldots, I_{n_k}\} \subseteq \{I_n\}$ such that $I \subseteq \cup_{i=1}^k I_{n_i}$. Let $I_{n_i} := (a_i, b_i)$ for $i \in \{1, \ldots, k\}$. Without loss of generality, assume that $J_i \cap [a, b] \neq \varnothing$ and $a_{i+1} < b_i$ for $i \in \{1, \ldots, k-1\}$. Then

$$\begin{aligned}\sum_{i=1}^k \ell(I_{n_i}) &= \sum_{i=1}^k (b_i - a_i) \\ &= b_k + \sum_{i=1}^{k-1} (b_i - a_{i+1}) - a_1 \geq b_k - a_1 \geq b - a.\end{aligned}$$

Thus,

$$b - a \leq \sum_{i=1}^k \ell(I_{n_i}) \leq \sum_i \ell(I_i) \leq m^*(I) + \varepsilon.$$

This is true for all $\varepsilon > 0$. Hence, $b-a \leq m^*(I)$. This completes the proof of $m^*(I) = b-a$.
Case 2: Suppose I is an interval of finite length with end points a and b with $a < b$. Then for sufficiently small $\varepsilon > 0$, we have $[a+\varepsilon, b-\varepsilon] \subseteq I \subseteq [a-\varepsilon, b+\varepsilon]$ so that

$$\begin{aligned}b - a - 2\varepsilon &= m^*([a+\varepsilon, b-\varepsilon]) \leq m^*(I) \\ &\leq m^*([a-\varepsilon, b+\varepsilon]) = b - a + 2\varepsilon.\end{aligned}$$

Since this is true for every $\varepsilon > 0$, it follows that $m^*(I) = b - a$.
Case 3: Suppose I is of infinite length. Then for every $M > 0$ there exists a closed interval I_M of length M such that $I_M \subseteq I$. Hence $M = m^*(I_M) \leq m^*(I)$. Thus, $M \leq m^*(I)$ for all $M > 0$ so that $m^*(I) = \infty$. □

COROLLARY 1.12. *Every non-degenerate interval is an uncountable set.*

1.3 Lebesgue Measurable Sets

Suppose A_1 and A_2 are disjoint subsets of $\mathbb{R}$. We may expect that

$$m^*(A_1 \cup A_2) = m^*(A_1) + m^*(A_2). \tag{1.1}$$

Is it true for any two disjoint sets A_1 and A_2?

Exercise 1.13. Prove that if (1.1) holds for any two disjoint sets A_1 and A_2, then

$$m^*\Big(\bigcup_{n=1}^\infty A_n\Big) = \sum_{n=1}^\infty m^*(A_n) \tag{1.2}$$

holds for any pairwise disjoint denumerable family $\{A_n\}_{n=1}^\infty$. ♢

Exercise 1.14. Prove that (1.2) need not hold for every denumerable disjoint family $\{A_n\}_{n=1}^\infty$.

Hint: On $[0,1]$ consider the relation $x \sim y \iff x - y \in \mathbb{Q}$.

- Show that $\sim$ is an equivalence relation on $[0, 1]$.

Let E be the subset of $[0,1]$ such that its intersection with each equivalence class is a singleton set. (Such a set E exists by using the *axiom of choice.*) Let $\{r_1, r_2, \ldots\} = \mathbb{Q} \cap [-1,1]$ and $E_n := E + r_n$ for $n \in \mathbb{N}$.

• Show that $\{E_n\}$ is a pairwise disjoint family.

Observe: $[0,1] \subseteq \bigcup_{n=1}^{\infty} E_n \subseteq [-1,2]$ so that $1 \le m^*\Big(\bigcup_{n=1}^{\infty} E_n\Big) \le 3$. Therefore, if (1.2) is true, we arrive at a contradiction. ♢

Exercise 1.15. There exist disjoint subsets A_1 and A_2 of $\mathbb{R}$ such that

$$m^*(A_1 \cup A_2) \ne m^*(A_1) + m^*(A_2).$$

Why? ♢

The above discussion motivates us to consider a family of sets in which the relations (1.2) holds for all possible pairwise disjoint denumerable family of sets involved.

Definition 1.16. A set $E \subseteq \mathbb{R}$ is said to be *Lebesgue measurable* if for every $A \subseteq \mathbb{R}$,

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c).$$

We denote the set of all Lebesgue measurable sets by $\mathcal{M}$. ♢

Exercise 1.17. Prove the following:

(i) $E \in \mathcal{M} \iff m^*(A) \ge m^*(A \cap E) + m^*(A \cap E^c) \quad \forall\, A \subseteq \mathbb{R}$.

(ii) $\varnothing \in \mathcal{M}$.

(iii) $E \in \mathcal{M} \Rightarrow E^c \in \mathcal{M}$.

(iv) $m^*(E) = 0 \Rightarrow E \in \mathcal{M}$. ♢

THEOREM 1.18. *Suppose $A_1 \cap A_2 = \varnothing$. If one of A_1, A_2 belongs to $\mathcal{M}$, then*

$$m^*(A_1 \cup A_2) = m^*(A_1) + m^*(A_2).$$

Proof. Suppose $A_1 \cap A_2 = \varnothing$. Assume $A_1 \in \mathcal{M}$. Then

$$m^*(A_1 \cup A_2) = m^*((A_1 \cup A_2) \cap A_1) + m^*((A_1 \cup A_2) \cap A_1^c).$$

But, $(A_1 \cup A_2) \cap A_1) = A_1$ and $(A_1 \cup A_2) \cap A_1^c) = A_2$. Hence the result □

Exercise 1.19. Prove that if $\{A_1, \ldots, A_n\}$ is a disjoint family in $\mathcal{M}$, then

$$m^*\Big(\bigcup_{i=1}^{n} A_i\Big) = \sum_{i=1}^{n} m^*(A_i).$$

[*Hint*: Use Theorem 1.18.] ♢

THEOREM 1.20. *If $\{A_n\}_{n=1}^{\infty}$ is a disjoint family in $\mathcal{M}$, then*

$$m^*\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} m^*(A_n).$$

Proof. Let $\{A_n\}_{n=1}^{\infty}$ be a disjoint family in $\mathcal{M}$. Then, we know that by Theorem 1.18, that

$$m^*(A_1 \cup A_2) = m^*(A_1) + m^*(A_2),$$

and hence,

$$m^*\Big(\bigcup_{i=1}^{n} A_i\Big) = \sum_{i=1}^{n} m^*(A_i) \qquad \forall n \in \mathbb{N}.$$

Thus, for every $n \in \mathbb{N}$,

$$\sum_{i=1}^{n} m^*(A_i) = m^*\Big(\bigcup_{i=1}^{n} A_i\Big) \le m^*\Big(\bigcup_{i=1}^{\infty} A_i\Big) \le \sum_{i=1}^{\infty} m^*(A_i)$$

so that

$$\sum_{i=1}^{n} m^*(A_i) \le m^*\Big(\bigcup_{i=1}^{\infty} A_i\Big) \le \sum_{i=1}^{\infty} m^*(A_i).$$

This is true for every $n \in \mathbb{N}$. Hence,

$$\sum_{i=1}^{\infty} m^*(A_i) \le m^*\Big(\bigcup_{i=1}^{\infty} A_i\Big) \le \sum_{i=1}^{\infty} m^*(A_i).$$

□

More importantly, we have the following theorem, whose proof we omit.

THEOREM 1.21. *If $E_n \in \mathcal{M}$ for $n \in \mathbb{N}$, then $\bigcup_n E_n \in \mathcal{M}$.*

Definition 1.22.
1. The outer measure m^* restricted to $\mathcal{M}$ is called the **Lebesgue measure** on $\mathbb{R}$, and it is denoted by m.
2. For $E \in \mathcal{M}$, $m(E) := m^*(E)$ is called the **Lebesgue measure of** E.

♢

Now the question is: How large the class $\mathcal{M}$ is?

THEOREM 1.23. *For any $a \in \mathbb{R}$, $(a, \infty) \in \mathcal{M}$.*

Proof. For $a \in \mathbb{R}$, let $E = (a, \infty)$. Let $A \subseteq \mathbb{R}$ and $\varepsilon > 0$. By Proposition **??**, there exists a sequence (I_n) of open intervals such that

$$A \subseteq \bigcup_n I_n, \qquad \sum_n \ell(I_n) < m^*(A) + \varepsilon.$$

Let $A_1 = A \cap (a, \infty)$ and $A_2 = A \cap (-\infty, a]$. Clearly, A is the disjoint union of A_1 and A_2. If we write $I_n' = I_n \cap (a, \infty)$ and $I_n'' = I_n \cap (-\infty, a]$, then we have I_n is the disjoint union of I_n' and I_n'', so that $\ell(I_n) = \ell(I_n') + \ell(I_n'')$. Also, $A_1 \subseteq \bigcup_n I_n'$, $A_2 \subseteq \bigcup_n I_n''$. Hence,

$$\begin{aligned} m^*(A_1) + m^*(A_2) &\le \sum_n \ell(I_n') + \sum_n \ell(I_n'') \\ &= \sum_n \ell(I_n) \le m^*(A) + \varepsilon. \end{aligned}$$

This is true for any $\varepsilon > 0$, so that we get

$$m^*(A_1) + m^*(A_2) \le m^*(A),$$

which completes the proof. $\square$

Using the Theorem 1.23 and the results which we have already listed, the following can be deduced.

- For any $a \in \mathbb{R}$, $[a, \infty) \in \mathcal{M}$.
- For any $a \in \mathbb{R}$, $(-\infty, a) \in \mathcal{M}$.
- For any $a \in \mathbb{R}$, $(-\infty, a] \in \mathcal{M}$.
- For any $a, b \in \mathbb{R}$ with $a < b$, $(a, b) \in \mathcal{M}$.
- For any $a, b \in \mathbb{R}$ with $a < b$, $[a, b] \in \mathcal{M}$.
- Every open subset of $\mathbb{R}$ belongs to $\mathcal{M}$.
- Every closed subset of $\mathbb{R}$ belongs to $\mathcal{M}$.
- Every G_δ subset of $\mathbb{R}$ belongs to $\mathcal{M}$.
- Every F_σ subset of $\mathbb{R}$ belongs to $\mathcal{M}$.

Is there any subset of $\mathbb{R}$ which does not belong to $\mathcal{M}$?

Exercise 1.24. There exists $E \subseteq \mathbb{R}$ such that $E \notin \mathcal{M}$ - Why?. $\diamondsuit$

Exercise 1.25. If $E \in \mathcal{M}$ with $m(E) < \infty$ then show that for every $\varepsilon > 0$, there exists an open set $G \subseteq \mathbb{R}$ such that $G \supseteq E$ and $m(G \setminus E) < \varepsilon$. $\diamondsuit$

Analogues to the definition of outer measure and Lebesgue measure on $\mathbb{R}$, we can define outer measure and Lebesgue measure on $\mathbb{R}^k$ for any $k \in \mathbb{N}$.

Definition 1.26.

1. For $E \subseteq \mathbb{R}^k$, the **Lebesgue outer measure** of E is defined as

$$m^*(E) := \inf_{\mathcal{D}} \sum_n vol(D_n),$$

where the infimum is taken over the set $\mathcal{D}$ of all countable collection $\{D_n\}$ of open parallelopipeds (or open rectangles if $k = 2$) such that $E \subseteq \cup_n D_n$, and $vol(D_n)$ denotes the volume (area if $k = 2$) of D_n .

2. A set $E \subseteq \mathbb{R}^k$ is said to be *Lebesgue measurable* if

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c) \quad \forall\, A \subseteq \mathbb{R}^k.$$

$\diamondsuit$

We denote the set of all Lebesgue measurable sets by $\mathcal{M}_k$.

It can be shown that $\mathcal{M}_k$ has properties analogous to $\mathcal{M}$.

2 Lecture 2: Measure and measurable functions

2.1 Measure on an Arbitrary σ-Algebra

Recall:

- $\mathbb{R} \in \mathcal{M}$,
- $E \in \mathcal{M}$ implies $E^c \in \mathcal{M}$,
- $E_n \in \mathcal{M}$ for $n \in \mathbb{N}$ implies $\bigcup_n E_n \in \mathcal{M}$, and
- $\{E_n\}$ is a countable disjoint family in $\mathcal{M}$ implies

$$m\Big(\bigcup_n E_n\Big) = \sum_n m(E_n).$$

Definition 2.1. Let X be a set and $\mathcal{A}$ be a family of subsets of X. Then $\mathcal{A}$ is called a **σ-algebra** if

(a) $X \in \mathcal{A}$,

(b) $A \in \mathcal{A}$ implies $A^c \in \mathcal{A}$,

(c) $A_n \in \mathcal{A}$ for $n \in \mathbb{N}$ implies $\bigcup_n A_n \in \mathcal{M}$.

The pair $(X, \mathcal{A})$ is called a **measurable space**, and members of a σ-algebra are called **measurable sets**.

$\diamondsuit$

Definition 2.2. Let $(X, \mathcal{A})$ be a measurable space. A function $\mu : \mathcal{A} \to [0, \infty]$ is called a **positive measure** or simply a **measure** if

(a) $\mu(\varnothing) < \infty$, and

(b) for any countable disjoint family $\{A_n\}$ in $\mathcal{A}$,

$$\mu\Big(\bigcup_n A_n\Big) = \sum_n \mu(A_n).$$

The pair $(X, \mathcal{A}, \mu)$ is called a **measure space**. For $A \in \mathcal{A}$, $\mu(A)$ is called the **measure of** A. $\diamondsuit$

According to the above definitions, $(\mathbb{R}, \mathcal{M})$ is a measurable space and $(\mathbb{R}, \mathcal{M}, m)$ is a measure space.

Let $(X, \mathcal{A}, \mu)$ be a measure space. Then the following results can be easily verified:

- For $A, B \in \mathcal{A}$, if $A \subseteq B$ then $\mu(A) \leq \mu(B)$.
- For $A, B \in \mathcal{A}$, if $A \subseteq B$ and $\mu(B) < \infty$ then

$$\mu(B \setminus A) = \mu(B) - \mu(A).$$

Example 2.3. Let X be any set, $\mathcal{A} = \{\varnothing, X\}$ and let μ be defined by $\mu(\varnothing) = 0$ and $\mu(X) = 1$. $\diamondsuit$

Example 2.4. Let X be any set, $\mathcal{A}$ be the power set of X, and μ be defined by

$$\mu(E) = \begin{cases} \#(E) & \text{if } E \text{ is a finite set} \\ \infty & \text{if } E \text{ is an infinite set.} \end{cases}$$

Here, $\#(E)$ denotes the cardinality of E. The above measure is called the **counting measure** on X. ♢

Example 2.5. Let X be any set, $\mathcal{A}$ be the power set of X, $x_0 \in X$, and μ be defined by

$$\mu(E) = \begin{cases} 1 & \text{if } x_0 \in E \\ 0 & \text{if } x_0 \notin E. \end{cases}$$

This measure is called the **Dirac measure** on X centered at x_0.
[Think of a *thin wire* of *negligible weight*. Suppose a bead of weight 1 unit kept at some point x_0 on the wire. Then weight of a part, say E, of the wire is 1 if x_0 belongs to E, and the wight is zero if x_0 does not belong to E.] ♢

More generally we have the following.

Example 2.6. Let X be a set, $\mathcal{A}$ be the power set of X, $x_i \in X$ and $w_i \geq 0$ for $i \in \{1, \ldots, k\}$. For $E \subseteq X$, let $\Delta_E = \{i : x_i \in E\}$, and let μ be defined by

$$\mu(E) = \sum_{i \in \Delta_E} w_i.$$

[Think of a *thin wire* of *negligible weight*. Suppose a finite number of beads of weights $w_1, \ldots, w_k$ are kept at points $x_1, \ldots, x_k$ respectively on the wire. Then the weight of a part, say E, of the wire is $\sum_{i \in \Delta_E} w_i$ where $\Delta_E = \{i : x_i \in E\}$.] ♢

THEOREM 2.7. *Suppose $\{\mathcal{A}_\alpha : \alpha \in \Lambda\}$ is a family of σ-algebras on a set X. Then $\bigcap_{\alpha \in \Lambda} \mathcal{A}_\alpha$ is also a σ-algebra.*

Proof. Exercise. □

Definition 2.8. Let X be a set and $\mathcal{S}$ be a family of subsets of X. Then the intersection of all σ-algebras containing $\mathcal{S}$ is called the **σ-algebra generated by $\mathcal{S}$**. ♢

Definition 2.9. Let X be a topological space with topology $\mathcal{T}$. Then the σ-algebra generated by $\mathcal{T}$ is called the **Borel σ-algebra** or **Borel field** on X.

♢

Definition 2.10. The **Borel σ-algebra** on $\mathbb{R}^k$ is denoted by $\mathcal{B}_k$. ♢

Since the σ-algebra $\mathcal{M}_k$ contains all open sets, it follows that $\mathcal{B}_k \subseteq \mathcal{M}_k$.

THEOREM 2.11. *Suppose* $(X, \mathcal{A})$ *is a measurable space and* $X_0 \in \mathcal{A}$. *Then*

$$\mathcal{A}_0 := \{E \subseteq X_0 : E \in \mathcal{A}\}$$

is a σ*-algebra on* X_0, *and*

$$\mathcal{A}_0 = \{A \cap X_0 : A \in \mathcal{A}\}.$$

Proof. Exercise □

- The σ-algebra $\mathcal{A}_0$ in Theorem 2.11 is called the **restriction of** $\mathcal{A}$ **to** X_0.
- Members of $\mathcal{A}_0$ are called **measurable subsets of** $\mathcal{A}_0$.
- If μ is a measure on $(X, \mathcal{A})$, then $\mu_0 := \mu|_{\mathcal{A}_0}$ is a measure on the measurable space $(X_0, \mathcal{A}_0)$.

Definition 2.12. A measure μ on a measurable space $(X, \mathcal{A})$ is called a **complete measure** if for every $A \in \mathcal{A}$ with $\mu(A) = 0$, we have $E \in \mathcal{A}$ for every $E \subseteq A$. ♢

- Lebesgue measure on $\mathcal{M}$ is complete.
- Lebesgue measure on the Borel σ-algebra is not complete.
  [Its proof requires the concept of *Lebesgue measurable functions* which we shall introduce shortly.
- If $\#(X) \geq 2$, and $\mathcal{A} = \{X, \varnothing\}$, and μ defined by $\mu(X) = 1$, $\mu(\varnothing) = 0$, then μ is not complete.

THEOREM 2.13. *Suppose* $A_i \in \mathcal{A}$ *such that*

$$A_n \subseteq A_{n+1} \qquad \forall\, n \in \mathbb{N}.$$

Then

$$\mu(A_n) \to \mu\left(\bigcup_{i=1}^{\infty} A_i\right) \quad \textit{as} \quad n \to \infty.$$

Proof. We write $\bigcup_{i=1}^{\infty} A_i$ as a disjoint union $\bigcup_{i=1}^{\infty} E_i$ by taking

$$E_1 = A_1, \quad E_i = A_i \setminus A_{i-1}, \quad i = 2, 3, \ldots.$$

Then

$$\begin{aligned} \mu\Big(\bigcup_{i=1}^{\infty} A_i\Big) &= \mu\Big(\bigcup_{i=1}^{\infty} E_i\Big) = \sum_{i=1}^{\infty} \mu(E_i) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \mu(E_i) = \lim_{n\to\infty} \mu\Big(\bigcup_{i=1}^{n} E_i\Big). \end{aligned}$$

But, $\bigcup_{i=1}^{n} E_i = A_n$. Hence,

$$\mu\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \lim_{n\to\infty} \mu(A_n).$$

This completes the proof. □

THEOREM 2.14. *Suppose $A_n \in \mathcal{A}$ such that*

$$A_n \supseteq A_{n+1} \quad \forall\, n \in \mathbb{N} \quad \textit{and} \quad \mu(A_1) < \infty.$$

Then

$$\mu(A_n) \to \mu\left(\bigcap_{i=1}^{\infty} A_i\right) \quad \textit{as} \quad n \to \infty.$$

Proof. We Theorem 2.13: Write $B_n = A_1 \setminus A_n$, $n \in \mathbb{N}$. Then $B_n \subseteq B_{n+1}$ for all $n \in \mathbb{N}$, so that by the above theorem,

$$\mu\Big(\bigcup_{i=1}^{\infty} B_i\Big) = \lim_{n\to\infty} \mu(B_n).$$

But,

$$\bigcup_{i=1}^{\infty} B_i = A_1 \setminus \bigcap_{i=1}^{\infty} A_i, \quad B_n = A_1 \setminus A_n.$$

Therefore, since $\mu(A_1) < \infty$, we have

$$\mu\Big(\bigcup_{i=1}^{\infty} B_i\Big) = \mu\Big(A_1 \setminus \bigcap_{i=1}^{\infty} A_i\Big) = \mu(A_1) - \mu\Big(\bigcap_{i=1}^{\infty} A_i\Big),$$

$$\mu(B_n) = \mu(A_1 \setminus A_n) = \mu(A_1) - \mu(A_n),$$

from which we obtain the required relations. □

Measurable Functions

Definition 2.15. Let $(X, \mathcal{A})$ be a measurable space. Then a function $f : X \to \mathbb{R}$ is said to be a **real measurable function** if $f^{-1}(G) \in \mathcal{A}$ for every open set G in $\mathbb{R}$. ♢

Similarly we can define a **complex measurable function** if the codomain of f is $\mathbb{C}$ instead of $\mathbb{R}$

Observe:

- If $X = E \subseteq \mathbb{R}$ and $\mathcal{A}$ is either $\mathcal{M}_E$ or $\mathcal{B}_E$, then every continuous function $f : E \to \mathbb{R}$ is measurable.

More generally:

Definition 2.16. Let $(X, \mathcal{A})$ be a measurable space and $(Y, \mathcal{T})$ be a topological space. Then a function $f : X \to Y$ is said to be a **measurable** with respect to the pair $(\mathcal{A}, \mathcal{T})$ if $f^{-1}(G) \in \mathcal{A}$ for every $G \in \mathcal{T}$.

♢

Exercise 2.17. Let $(X, \mathcal{A})$ be a measurable space. Show that a real valued function $f : X \to \mathbb{R}$ is measurable if and only if $f^{-1}(I) \in \mathcal{A}$ for every open interval I. ♢

Some of the topological spaces that we frequently use in the study of measure and integration are the following:

- $\mathbb{R}$ with usual topology,
- $\mathbb{C}$ with usual topology,
- $\mathbb{R}^k$ with usual topology.
- $\tilde{\mathbb{R}} := [-\infty, \infty]$ with topology $\mathcal{T}$ consisting of unions of intervals of the form (a, b), $(a, \infty]$, $[-\infty, b)$.

THEOREM 2.18. *Let X be a measurable space, and Y and Z be topological spaces. If $f : X \to Y$ is measurable and $g : Y \to Z$ is continuous, then $g \circ f : X \to Z$ is measurable.*

Proof. Obvious from the definition. □

THEOREM 2.19. *Let X be a measurable space and Y be a topological space. If u, v are real measurable functions on X and $g : \mathbb{R}^2 \to Y$ is continuous, then the function $\varphi : X \to Y$ defined by*

$$\varphi(x) = g(u(x), v(x)), \qquad x \in X,$$

is a measurable function from X to Y.

Proof. Define $f : X \to \mathbb{R}^2$ by $f(x) = (u(x), v(x))$, $x \in X$. Then $\varphi = g \circ f$. Since g is continuous, by the last theorem, it is enough to prove that f is a measurable function. For this it is sufficient to show that for open intervals I_1 and I_2, $f^{-1}(I_1 \times I_2)$ is measurable. Note that

$$x \in f^{-1}(I_1 \times I_2) \iff f(x) \in I_1 \times I_2$$

$$\iff u(x) \in I_1 \,\&\, v(x) \in I_2 \iff x \in u^{-1}(I_1) \cap v^{-1}(I_2).$$

Thus $f^{-1}(I_1 \times I_2) = u^{-1}(I_1) \cap v^{-1}(I_2)$ which is measurable. □

The following two theorems can be deduced from the above theorems.

THEOREM 2.20. *Let X be a measurable space.*

(i) If u, v are real measurable functions on X, then $u + iv$ is a complex measurable function on X.

(ii) If f is a complex measurable function on X, then $\mathrm{Re}f$, $\mathrm{Im}f$ and $|f|$ are real measurable functions on X.

THEOREM 2.21. *Let X be a measurable space. If f and g are real (resp. complex) measurable functions on X, then $f \pm g$ and fg are real (resp. complex) measurable functions on X.*

THEOREM 2.22. *Let X be a measurable space.*

(i) For $E \subseteq X$, χ_E is measurable if and only if E is measurable.

(ii) If $f : X \to \mathbb{C}$ is measurable, then there exists a measurable function $\alpha : X \to \mathbb{C}$ such that $|\alpha| = 1$ and $f = \alpha|f|$.

Proof. (i) Let $E \subseteq X$. For an open set G,

$$\chi_{_E}^{-1}(G) = \begin{cases} E, & 1 \in G,\, 0 \notin G, \\ E^c, & 1 \notin G,\, 0 \in G, \\ X, & 1 \in G,\, 0 \in G, \\ \varnothing, & 1 \notin G,\, 0 \notin G. \end{cases}$$

Thus, $\chi_{_E}$ is measurable if and only if E is measurable.

(ii) Let

$$\alpha(x) := \begin{cases} \frac{f(x)}{|f(x)|}, & f(x) \neq 0, \\ 1, & f(x) = 0. \end{cases}$$

That is, if $E = \{x : f(x) = 0\}$, then

$$\alpha(x) = \frac{f(x) + \chi_{_E}(x)}{|f(x) + \chi_{_E}(x)|} = \Phi(f(x) + \chi_{_E}(x), \quad x \in X,$$

where $\Phi(z) := z/|z|$ for $z \in \mathbb{C} \setminus \{0\}$. Since Φ is continuous and f and $\chi_{_E}$ are measurable, α is measurable. Also, $f(x) = \alpha(x)|f(x)|$ and $|\alpha(x)| = 1$ for all $x \in X$. □

In the following $(X, \mathcal{A})$ is a measurable space.

THEOREM 2.23. *Let $E \subseteq X$. Then $E \in \mathcal{A}$ if and only if χ_E is a measurable function.*

THEOREM 2.24. *Let $(Y, \mathcal{T})$ be a topological space.*

(i) *$\mathcal{S} := \{E \subseteq Y : f^{-1}(E) \in \mathcal{A}\}$ is a σ-algebra on Y.*

(i) *If f is measurable, then $\mathcal{S}$ contains the Borel field on Y.*

As a corollary to the above theorem, we have the following.

THEOREM 2.25. *Let $(Y, \mathcal{T})$ be a topological space. Then $f : X \to Y$ is measurable if and only if for every Borel set A in Y, $f^{-1}(A)$ is measurable in X.*

Remark 2.26. Suppose $(X_1, \mathcal{A}_1)$ and $(X_2, \mathcal{A}_2)$ are two measurable spaces. Then we may define a function $f : X_1 \to X_2$ to be measurable if $f^{-1}(A) \in \mathcal{A}_1$ for every $A \in \mathcal{A}_2$. Then, in view of the above theorem, our definition becomes a particular case by taking X_2 a topological space and $\mathcal{A}_2$ the Borel filed on X_2. ♢

The following theorem can be verified easily.

THEOREM 2.27. *For a function $f : X \to [-\infty, \infty]$, the following are equivalent:*

(i) *f is measurable.*

(ii) $\{x \in X : f(x) > a\} \in \mathcal{A} \quad \forall a \in \mathbb{R}.$

(iii) $\{x \in X : f(x) \geq a\} \in \mathcal{A} \quad \forall a \in \mathbb{R}.$

(iv) $\{x \in X : f(x) < a\} \in \mathcal{A} \quad \forall a \in \mathbb{R}.$

(v) $\{x \in X : f(x) \leq a\} \in \mathcal{A} \quad \forall a \in \mathbb{R}.$

THEOREM 2.28. *Let $f_n : X \to [-\infty, \infty]$ be measurable for each $n \in \mathbb{N}$. Then*

$$\sup_n f_n, \quad \inf_n f_n, \quad \limsup_n f_n, \quad \liminf_n f_n$$

are measurable functions.

Proof. For $\alpha \in \mathbb{R}$, we note that

$$\{x \in X : \sup_n f_n > \alpha\} = \bigcup_n \{x \in X : f_n > \alpha\},$$

$$\{x \in X : \inf_n f_n < \alpha\} = \bigcup_n \{x \in X : f_n < \alpha\}.$$

From this it follows that $\sup_n f_n$ and $\inf_n f_n$ are measurable functions. Since

$$\limsup_n f_n = \inf_k \sup_{n \geq k} f_n, \quad \liminf_n f_n = \sup_k \inf_{n \geq k} f_n$$

it also follows that $\limsup_n f_n$ and $\liminf_n f_n$ are measurable functions. □

COROLLARY 2.29. *Let (f_n) be a sequence of extended real valued (resp. complex valued) measurable functions on X. The following hold:*

(i) *If $f_n(x) \to f(x)$ as $n \to \infty$ for each $x \in X$, then f is measurable.*

(ii) *If $\sum_{n=1}^{\infty} f_n(x)$ converges for each $x \in X$ and $f(x) = \sum_{n=1}^{\infty} f_n(x)$, then f is measurable.*

THEOREM 2.30. *Let $f : X \to [-\infty, \infty]$. If f is measurable, then f^+ and f^- are measurable. In case f is real valued, then f is measurable if and only if f^+ and f^- are measurable, and in that case $|f|$ is also measurable.*

Proof. Note that $f^+(x) := \max\{f(x), 0\}$, $f^-(x) := -\min\{f(x), 0\}$, and $|f| = f^+ + f^-$. Hence, the proof follows from Theorem 2.28. □

Definition 2.31. A function $\varphi : X \to \mathbb{R}$ is called a **simple function** if it takes only a finite number of values.

Suppose $\varphi : X \to \mathbb{R}$ is a simple function taking distinct values $\alpha_1, \ldots, \alpha_n$. Then $\varphi = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$, where $E_i = \{x \in X : \varphi(x) = \alpha_i\}$, $i = 1, \ldots, n$. The above representation of φ is called its **canonical representation**, and $X = \cup_{i=1}^{n} E_i$. ♢

It is obvious that $\varphi : X \to \mathbb{R}$ is a simple function if and only if there exist $a_1, \ldots, a_n$ in $\mathbb{R}$ and $A_i \subseteq X$, $i = 1, \ldots, n$, such that

$$\varphi = \sum_{i=1}^{n} a_i \chi_{A_i}.$$

Exercise 2.32. Let $(X, \mathcal{A})$ be a measurable space and φ is a simple function on X with canonical representation $\varphi = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$. Then φ is measurable if and only if $E_i \in \mathcal{A}$ for $i = 1, \ldots, n$ – Justify. $\diamondsuit$

THEOREM 2.33. *Let $(X, \mathcal{A})$ be a measurable space and $f : X \to [0, \infty]$ be a measurable function. For each $n \in \mathbb{N}$, let*

$$\begin{aligned} E_{i,n} &:= \left\{ x \in X : \frac{i-1}{2^n} \le f(x) < \frac{i}{2^n} \right\}, \quad i = 1, \ldots, n2^n, \\ F_n &:= \{x \in X : f(x) \ge n\}, \\ \varphi_n &= \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \chi_{E_{i,n}} + n\chi_{F_n}. \end{aligned}$$

Then each φ_n is measurable, $0 \le \varphi_n \le \varphi_{n+1}$ for every $n \in \mathbb{N}$ and $\varphi_n(x) \to f(x)$ as $n \to \infty$ for each $x \in X$.

Proof. Let $x \in X$. If $f(x) = \infty$, then $\varphi_n(x) = n$ for all $n \in \mathbb{N}$, so that $\varphi_n(x) \to f(x)$ as $n \to \infty$. In case $f(x) \in \mathbb{R}$, then there exists $k \in \mathbb{N}$ such that $f(x) \le k$, so that $x \in E_{i,n}$ for all $n > k$ and for some $i \in \{1, 2, \ldots, n2^n\}$, and in that case $|f(x) - \varphi_n(x)| \le 1/2^n$. Hence, in this case also, $\varphi_n(x) \to f(x)$ as $n \to \infty$.

Next suppose that $x \in E_{i,n}$ for some $n \in \mathbb{N}$ and for some $i \in \{1, 2, \ldots, n2^n\}$. Then $\varphi_n(x) = (i-1)/2^n$, and

$$\varphi_{n+1}(x) \in \left\{ \frac{i-1}{2^n}, \; \frac{i-1}{2^n} + \frac{1}{2^{n+1}} \right\}.$$

Thus, $\varphi_n(x) \le \varphi_{n+1}(x)$. If $x \in F_n$, then $\varphi_n(x) = n$ and $\varphi_{n+1}(x) \in \{n, \; n + \frac{1}{2^{n+1}}\}$. Thus, we get $\varphi_n(x) \le \varphi_{n+1}(x)$ for every $x \in X$ and for every $n \in \mathbb{N}$. $\square$

According to the above theorem, every extended real valued non-negative measurable function f can be approximated pointwise by an increasing sequence (φ_n) of simple measurable functions. Thus, if $(X, \mathcal{A})$ is endowed with a measure μ, then a natural procedure of defining the integral of a extended real valued non-negative measurable function f would be to define the integral $\int_X \varphi d\mu$ of simple measurable functions φ, and then define $\int_X f d\mu$ as the limit (if exists) of the sequence $\left(\int_X \varphi_n d\mu\right)$ whenever (φ_n) is an increasing sequence of non-negative simple measurable functions which converges to f pointwise. But, to have such an integral to be well-defined, we must also show the following:

- If (φ_n) and (ψ_n) are increasing sequence of non-negative simple measurable functions which converge to f pointwise, then the limits $\lim_{n\to\infty} \int_X \varphi_n d\mu$ and $\lim_{n\to\infty} \int_X \psi_n d\mu$ exist and must be equal.

We shall do this in the next chapter.

3 Lecture 3: Integral of measurable functions

3.1 Integral of a Simple Measurable Function

Definition 3.1. Let $(X, \mathcal{A}, \mu)$ be a measure space. Let φ be a non-negative simple measurable function with the *canonical representation* $\varphi := \sum_{i=1}^{n} \alpha_i \chi_{A_i}$, i.e., $\alpha_1, \ldots, \alpha_n$ are distinct values of φ and $A_i := \{x \in X : \varphi(x) = \alpha_i\}$ for $i \in \{1, \ldots, n\}$. Then we define

$$\int_X \varphi d\mu := \sum_{i=1}^{n} \alpha_i \mu(A_i)$$

and it is called the **integral of** φ **over** X with respect to μ. ♢

Remark 3.2. In the above definition of the integral we assumed φ to be non-negative, because otherwise in the sum $\sum_{i=1}^{n} \alpha_i \mu(A_i)$ may not be well-defined since it can happen that $\mu(A_i) = \mu_j(A_j) = \infty$ for some $i \neq j$ and $\alpha_i < 0 < \alpha_j$. ♢

It is easy to see that if $\varphi = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$ is the canonical representation of a simple measurable function φ, then for any $E \in \mathcal{A}$, $\chi_E \varphi = \sum_{i=1}^{n} \alpha_i \chi_{A_i \cap E}$ is the canonical representation of $\chi_E \varphi$ so that

$$\int_X \chi_E \varphi d\mu = \sum_{i=1}^{n} \alpha_i \mu(A_i \cap E).$$

Definition 3.3. If φ is a non-negative simple measure function on a measure space $(X, \mathcal{A}, \mu)$ and if $E \in \mathcal{A}$, then we define

$$\int_E \varphi d\mu := \int_X \chi_E \varphi d\mu,$$

and it called the **integral of** φ **over** E with respect to μ. ♢

We observe that

- A function $\varphi : X \to [0, \infty)$ is simple measurable if and only if there exists $\alpha_1, \ldots, \alpha_n$ in $[0, \infty)$ and measurable sets $E_1, \ldots, E_n$ such that $\varphi = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$.

A natural question is whether $\int_X \varphi d\mu = \sum_{i=1}^{n} \alpha_i \mu(E_i)$. The answer is in affirmative:

THEOREM 3.4. *Suppose $B_1, \ldots, B_k$ are measurable sets, $\beta_1, \ldots, \beta_k$ are non-negative real numbers and $\psi := \sum_{i=1}^{k} \beta_i \chi_{B_i}$. Then*

$$\int_X \psi d\mu = \sum_{i=1}^{k} \beta_i \mu(B_i).$$

Proof. *Case (i)*: $B_1, \ldots, B_k$ are disjoint:

In case $\beta_1, \ldots, \beta_k$ are distinct, then the given representation of ψ is its canonical representation and hence the proof follows from the definition.

Next assume that some of $\beta_1, \ldots, \beta_k$ are repeated. Suppose $\alpha_1, \ldots, \alpha_n$ are the distinct numbers among $\beta_1, \ldots, \beta_k$. For each $i \in \{1, \ldots, n\}$, let

$$\Delta_i := \{j : \beta_j = \alpha_i\}.$$

Then it is clear that

$$\psi = \sum_{i=1}^{k} \beta_i \chi_{B_i} = \sum_{i=1}^{n} \alpha_i \sum_{j \in \Delta_i} \chi_{B_j} = \sum_{i=1}^{n} \alpha_i \chi_{A_i},$$

where $A_i = \bigcup_{j \in \Delta_i} B_j$. Hence,

$$\begin{aligned}\int_X \psi d\mu &= \sum_{i=1}^{n} \alpha_i \mu(A_i) = \sum_{i=1}^{n} \alpha_i \sum_{j \in \Delta_i} \mu(B_j) \\ &= \sum_{i=1}^{n} \sum_{j \in \Delta_i} \beta_j \mu(B_j) = \sum_{i=1}^{k} \beta_i \mu(B_i).\end{aligned}$$

Case (ii): $B_1, \ldots, B_k$ are not necessarily disjoint:

In this case first we may write $\bigcup_{i=1}^{k} B_i$ as a disjoint union $\bigcup_{j=1}^{k} C_i$. Then

$$B_i = B_i \cap \bigcup_{j=1}^{k} C_j = \bigcup_{j=1}^{k} (B_i \cap C_j)$$

so that $\chi_{B_i} = \sum_{j=1}^{k} \chi_{B_i \cap C_j}$ and hence

$$\psi := \sum_{i=1}^{k} \beta_i \sum_{j=1}^{k} \chi_{B_i \cap C_j} = \sum_{i=1}^{k} \sum_{j=1}^{k} \beta_i \chi_{B_i \cap C_j}.$$

This representation of ψ fits into the case (i) discussed above. □

THEOREM 3.5. *If φ and ψ are non-negative simple measurable functions and $c \in \mathbb{R}$, then*

$$\int_X (\varphi + \psi) d\mu = \int_X \varphi d\mu + \int_X \psi d\mu,$$

$$\int_X c\,\varphi d\mu = c \int_X \varphi d\mu.$$

Proof. Suppose $\varphi := \sum_{i=1}^{n} \alpha_i \chi_{A_i}$ and $\psi := \sum_{j=1}^{m} \beta_j \chi_{B_j}$ are the canonical representations of φ and ψ respectively. Then $\varphi + \psi$ and $c\,\varphi$ have the representations

$$\varphi + \psi = \sum_{i=1}^{n} \alpha_i \chi_{A_i} + \sum_{j=1}^{m} \beta_j \chi_{B_j}, \quad c\,\varphi = \sum_{i=1}^{n} c\,\alpha_i \chi_{A_i}$$

respectively. Hence, the proof follows by applying last theorem. □

THEOREM 3.6. *If φ and ψ are non-negative simple measurable functions such that $\varphi \le \psi$, then*

$$\int_X \varphi d\mu \le \int_X \psi d\mu.$$

Proof. Since $\psi = \varphi + (\psi - \varphi)$ and $\psi - \varphi \ge 0$ and since $\int_X (\psi - \varphi) d\mu \ge 0$, by the last theorem,

$$\int_X \psi d\mu = \int_X \varphi d\mu + \int_X (\psi - \varphi) d\mu \ge \int_X \varphi d\mu.$$

This competes the proof. □

THEOREM 3.7. *Suppose $\varphi : X \to [0, \infty)$ is a simple measurable function. For $E \in \mathcal{A}$, let*

$$\nu(E) := \int_E \varphi d\mu.$$

Then ν is a measure on X.

Proof. Clearly $\nu(\varnothing) = 0$. Suppose $\varphi = \chi_A$ for some $A \in \mathcal{A}$. Then for $E \in \mathcal{A}$,

$$\nu(E) = \int_E \varphi d\mu = \mu(A \cap E).$$

If $\{E_n\}$ is a countable disjoint family in $\mathcal{A}$, then

$$\begin{aligned} \nu(\cup_{n=1}^{\infty} E_n) = \mu(A \cap (\cup_{n=1}^{\infty} E_n)) &= \mu(\cup_{n=1}^{\infty} (A \cap E_n)) \\ &= \sum_{n=1}^{\infty} \mu(A \cap E_n) = \sum_{n=1}^{\infty} \nu(E_n). \end{aligned}$$

Since a simple function is a finite linear combination of characteristic functions, the result follows by using Theorem 3.5. □

Exercise 3.8. Suppose φ is a non-negative simple measurable function and E is a measurable set. Let μ_E be the restriction of μ to the restricted σ-algebra $\mathcal{A}_E := \{A \subseteq E : A \in \mathcal{A}\}$. Show that $\int_E \varphi d\mu = \int_E \varphi d\mu_E$. ♢

3.2 Integral of Positive Measurable Functions

Let $(X, \mathcal{A}, \mu)$ be a measure space and $f : X \to [0, \infty]$ be a measurable function. We know that there exists a sequence (φ_n) of non-negative simple measurable functions such that

- $0 \le \varphi_n \le \varphi_{n+1} \le f \quad \forall n \in \mathbb{N}$,
- $\varphi_n \to f$ pointwise.

Also we know that

- $0 \le \int_X \varphi_n d\mu \le \int_X \varphi_{n+1} d\mu \quad \forall n \in \mathbb{N}$

so that $\{\int_X \varphi_n d\mu\}$ converges. Hence, we expect that our definition of $\int_X f d\mu$ should be such that

$$\int_X f d\mu = \lim_{n\to\infty} \int_X \varphi_n d\mu.$$

THEOREM 3.9. *Let $(X, \mathcal{A}, \mu)$ be a measure space and $f : X \to [0, \infty]$ be a measurable function. Let (φ_n) be an increasing sequence of non-negative simple measurable functions which converge to f pointwise. Then the limit $\lim_{n\to\infty} \int_X \varphi_n d\mu$ exists and*

$$\lim_{n\to\infty} \int_X \varphi_n d\mu = \sup_{\varphi\in\Phi_f} \int_E \varphi d\mu,$$

where Φ_f is the set of all simple measurable functions φ such that $0 \leq \varphi \leq f$.

Proof. By Theorem 3.6, $\int_X \varphi_n \leq \int \varphi_{n+1}$ for all $n \in \mathbb{N}$. Hence $\left(\int_X \varphi_n d\mu\right)$ converges in $[0, \infty]$. Let

$$\alpha := \lim_{n\to\infty} \int_X \varphi_n d\mu \quad \text{and} \quad \beta := \sup_{\varphi\in\Phi_f} \int_E \varphi d\mu.$$

Since $\varphi_n(x) \to \varphi(x)$ as $n \to \infty$ for every $x \in X$ we also have $\varphi_n \leq f$ for all $n \in \mathbb{N}$. Hence, $\int_X \varphi_n d\mu \leq \beta$ for all $n \in \mathbb{N}$ so that $\alpha \leq \beta$.

Now, it remains to prove that $\beta \leq \alpha$. That is to show that $\int_X \varphi \leq \alpha$ for every $\varphi \in \Phi_f$. So, let $\varphi \in \Phi_f$. We are going to show that

$$\int_X r\,\varphi \leq \alpha \;\forall\, r \in (0, 1), \tag{$*$}$$

so that by taking limit as $r \to 1$, the result will follow. Hence, let $r \in (0, 1)$. We claim that

$$X = \bigcup_{n=1}^{\infty} \{x \in X : r\varphi(x) \leq \varphi_n(x)\} = \bigcup_{n=1}^{\infty} E_n,$$

where $E_n = \{x \in X : r\varphi(x) \leq \varphi_n(x)\}$, $n \in \mathbb{N}$. To see this, note that for $x \in X$, if $f(x) = 0$, then we have $\varphi(x) = 0$ so that $x \in E_1$. If $f(x) > 0$, then $r\,\varphi(x) \leq rf(x) < f(x)$ so that, since $\varphi_n(x) \to f(x)$ as $n \to \infty$, there exists $k \in \mathbb{N}$ such that $r\,\varphi(x) \leq \varphi_k(x) \leq f(x)$. Hence, $x \in E_k$. Also, we have $E_n \subseteq E_{n+1}$ for all $n \in \mathbb{N}$. Now,

$$\int_X \varphi_n \geq \int_{E_n} \varphi_n \geq \int_{E_n} r\,\varphi = r\,\nu(E_n),$$

where $\nu : X \to [0, \infty]$ is defined by $\nu(E) = \int_E \varphi d\mu$ for $E \in \mathcal{A}$. We know, by Theorem 3.7, ν is a measure. Hence, by Theorem 2.13, that

$$\nu\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \lim_{n\to\infty} \nu(E_n).$$

Thus,

$$r \int_X \varphi = r\,\nu(X) = r\,\nu\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \lim_{n\to\infty} r\,\nu(E_n) \leq \lim_{n\to\infty} \int_X \varphi_n = \alpha.$$

Thus $(*)$ is proved, which completes the proof. □

Motivated by the above theorem we have the following definition.

Definition 3.10. Let $(X, \mathcal{A}, \mu)$ be a measure space and $f : X \to [0, \infty]$ be a measurable function. Then we define the integral of f over $E \in \mathcal{A}$ as

$$\int_E f d\mu := \sup_{\varphi \in \Phi_f} \int_E \varphi d\mu,$$

where Φ_f is the set of all simple measurable functions φ satisfying $0 \le \varphi \le f$. $\diamondsuit$

Once the measure μ under consideration is understood, we may write the integral $\int_E f d\mu$ as $\int_E f$ for every $E \in \mathcal{A}$, and we write $\int f$ for $\int_X f d\mu$.

Now we list some of the properties of the integral in the following theorem.

THEOREM 3.11. *Suppose that f, g are extended real valued non-negative measurable functions defined on a measure space X, $c \in \mathbb{R}$ and E is a measurable set. Then we have the following.*

(i) $f \le g \Rightarrow \int_E f \le \int_E g$ (ii) $A, B \in \mathcal{A}, A \subseteq B \Rightarrow \int_A f \le \int_B f$.
(iii) $\int_E c f = c \int_E f$, (iv) $f(x) = 0 \;\forall\, x \in E \Rightarrow \int_E f = 0$.
(v) $\mu(E) = 0 \Rightarrow \int_E f = 0$, (vi) $\int_E f = \int_X \chi_E f$.

Proof. (i) Suppose $\varphi \in \Phi_f$. Then $\varphi \in \Phi_g$, so that $\int_E \varphi \le \int_E g$. This implies that

$$\int_E f := \sup_{\varphi \in \Phi_f} \int_E \varphi \le \int_E g.$$

(ii) Suppose $A, B \in \mathcal{A}$ such that $A \subseteq B$, and $\varphi \in \Phi_f$. Then, clearly, $\int_A \varphi \le \int_B \varphi \le \int_B f$. Hence,

$$\int_A f := \sup_{\varphi \in \Phi_f} \int_A \varphi \le \int_E g.$$

(iii) Suppose $\varphi \in \Phi_f$. We know that $\int_E c\varphi = c \int_E \varphi$. Taking supremum over all $\varphi \in \Phi_f$, we get

$$\int_E c f = c \int_E f.$$

(iv) Suppose $\varphi \in \Phi_f$. If $f(x) = 0$ for all $x \in E$, then $\varphi(x) = 0$ for all $x \in E$ so that $\int_E \varphi = 0$. Hence, Taking supremum over all $\varphi \in \Phi_f$, we get $\int_E f = 0$.

(v) Suppose $\varphi \in \Phi_f$. If $\mu(E) = 0$, then it is clear that $\int_E \varphi = 0$ for all $\varphi \in \Phi_f$. Hence, $\int_E f = 0$.

(vi) Suppose $\varphi \in \Phi_f$. Then $\chi_E \varphi \le \chi_E f$ so that by (i) we have $\int_E \varphi = \int_X \chi_E \varphi \le \int_X \chi_E f$. Hence, $\int_E \varphi \le \int_X \chi_E f$. This is true for all $\varphi \in \Phi_f$. Therefore

$$\int_E f \le \int_X \chi_E f.$$

Next, suppose $\varphi \in \Phi_{\chi_E f}$. Then $\varphi \le \chi_E f \le f$ and $\int_E \varphi = \int_X \varphi$. Hence, $\int_X \varphi = \int_E \varphi \le \int_E f$ This is true for all $\varphi \in \Phi_{\chi_E f}$. Therefore,

$$\int_X \chi_E f \le \int_E f.$$

This completes the proof. $\square$

3.3 Convergence Theorems

THEOREM 3.12. (Monotone convergence theorem) *Suppose (f_n) is a sequence of extended real valued non-negative measurable functions on a measure space $(X, \mathcal{A}, \mu)$ such that $f_n \le f_{n+1}$ for all $n \in \mathbb{N}$ and $(f_n(x))$ converges for every $x \in X$. Then $f : X \to [0, \infty]$ defined by $f(x) = \lim_{n\to\infty} f_n(x)$, $x \in X$, is a measurable function on X and*

$$\lim_{n\to\infty} \int_X f_n = \int f.$$

Proof. By by Corollary 2.29, f is measurable, and by Theorem 3.11(i),

$$\int_X f_n \le \int f_{n+1} \le \int_X f \qquad \forall\, n \in \mathbb{N}.$$

Also, there exists $\alpha \in [0, \infty]$ such that $\int_X f_n \to \alpha$ as $n \to \infty$ so that $\alpha \le \int_X f$.

Now, it remains to prove that $\int_X f \le \alpha$. This part of the proof is exactly same as in Theorem 3.9 with f_n in place of φ_n. □

COROLLARY 3.13. *Suppose f and g are extended real valued non-negative measurable functions on a measure space $(X, \mathcal{A}, \mu)$ such that $f \le g$. Then*

$$\int_E (f + g) = \int_E f + \int_E g \quad \forall\, E \in \mathcal{A}.$$

Proof. Use Theorem 2.33, Theorem 3.5 and Theorem 3.12. □

COROLLARY 3.14. *Suppose (f_n) is a sequence of extended real valued non-negative measurable functions on a measure space $(X, \mathcal{A}, \mu)$ and $f := \sum_{n=1}^{\infty} f_n$. Then*

$$\int_X f = \sum_{n=1}^{\infty} \int f_n.$$

Proof. Use Corollary 3.13 and Theorem 3.12, □

COROLLARY 3.15. *Suppose $f : X \to [0, \infty)$ is a measurable function. For $E \in \mathcal{A}$, let*

$$\nu(E) := \int_E f d\mu.$$

Then ν is a measure on X. More over, for any measurable function $g : X \to [0, \infty)$,

$$\int_X g d\nu = \int_X g f d\mu.$$

Proof. Clearly, $\nu(\varnothing) = 0$. Suppose $\{E_n\}$ is a disjoint countable family in $\mathcal{A}$. Then,

$$\nu(\bigcup_i E_i) := \int_{\bigcup_i E_i} f d\mu = \int_X \chi_{\bigcup_i E_i} f d\mu.$$

Now, we observe that

$$\chi_{\bigcup_i E_i} f = \lim_{n\to\infty} \chi_{\bigcup_{i=1}^n E_i} f = \lim_{n\to\infty} \sum_{i=1}^{n} \chi_{E_i} f,$$

where $\left(\sum_{i=1}^{n} \chi_{E_i} f\right)$ is an increasing sequence of non-negative measurable functions. Hence by Monotone Convergence Theorem,

$$\int_X \chi_{\bigcup_i E_i} f d\mu = \lim_{n\to\infty} \int_X \sum_{i=1}^{n} \chi_{E_i} f = \lim_{n\to\infty} \sum_{i=1}^{n} \int_X \chi_{E_i} f = \sum_{i=1}^{\infty} \int_X \chi_{E_i} f.$$

Thus,

$$\nu(\bigcup_i E_i) = \int_X \chi_{\bigcup_i E_i} f d\mu = \sum_{i=1}^{\infty} \int_X \chi_{E_i} f = \sum_{i=1}^{\infty} \nu(E_i).$$

This completes the first part of the theorem. For the second part, first we observe that the relation $\int_X g d\nu = \int g f d\mu$ holds if g is a characteristic function of a measurable set, and hence it holds for all simple non-negative measurable functions as well. Since any measurable function $g : X \to [0,\infty)$ is a point-wise limit of an increasing sequence of simple non-negative measurable functions, the proof can be completed by invoking Monotone Convergence Theorem. □

Remark 3.16. The relation $\int_X g d\nu = \int g f d\mu$ usually written as

$$d\nu = f d\mu \quad \text{or} \quad \frac{d\nu}{d\mu} = f,$$

and f is called the (Radon-Nikodym) derivative of ν with respect to μ.

Note that if μ and ν are as in Corollary 3.15, then $\nu(E) = 0$ whenever $\mu(E) = 0$. A question naturally arises would be the following:

> If μ and ν are measures such that $\nu(E) = 0$ whenever $\mu(E) = 0$, then does there exists a non-negative measurable function f such that $\nu(E) = \int_E f d\mu$ for all $E \in \mathcal{A}$?

The answer is in affirmative when μ is a *σ-finte measure* , that is, if there exists a countable family $\{X_n\}$ of measurable sets such that $X = \cup X_n$ and $\mu(X_n) < \infty$ for every $n \in \mathbb{N}$. This result is known as the *Radon-Nikodym theorem.* ♢

COROLLARY 3.17. (Fatou's lemma) *Suppose (f_n) is a sequence of extended real valued non-negative measurable functions on a measure space $(X, \mathcal{A}, \mu)$. Then*

$$\int_X \left(\liminf_n f_n\right) \le \liminf_n \int f.$$

Proof. For each $k \in \mathbb{N}$, let $g_k = \inf_{n\ge k} f_n$. Then $g_k \le g_{k+1}$ for all $k \in \mathbb{N}$, and $\lim_{k\to\infty} g_k = f := \liminf_n f_n$. Hence by Monotone Convergence Theorem, $\lim_{k\to\infty} \int_X g_k = \int_X f$. But, since $g_k \le f_k$ for all $k \in \mathbb{N}$, we have $\int_X g_k \le \int_X f_k$ so that

$$\int_X f = \lim_{k\to\infty} \int_X g_k = \liminf_k \int_X g_k \le \liminf_k \int_X f_k.$$

This completes the proof. □

Example 3.18. Let $X = \{x_1, \ldots, x_k\}$ with the counting measure μ on the σ-algebra 2^X. Let $f : X \to [0, \infty)$. Then

$$f(x) = \sum_{i=1}^{k} f(x_i)\chi_{\{x_i\}}(x), \quad x \in X$$

and

$$\int_X f d\mu = \sum_{i=1}^{k} f(x_i)\mu(x_i) = \sum_{i=1}^{k} f(x_i).$$

♢

Exercise 3.19. Suppose $X = \{x_1, x_2, \ldots\}$ with the counting measure μ. If f is an extended real valued non-negative measurable function on X, then show that

$$\int_X f d\mu = \sum_{i=1}^{\infty} f(x_i).$$

♢

Exercise 3.20. Suppose $a_{ij} \geq 0$ for all $i, j \in \mathbb{N}$. Then show that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}.$$

♢

Exercise 3.21. Suppose $X = \{x_1, \ldots, x_n\}$, and $w_1, \ldots, w_n$ are non-negative reals. For $x \in X$, define $w(x) = w_j$ whenever $x = x_j$, and $\mu(E) = \sum_{x \in E} w(x)$ for $E \subseteq X$. Show that μ is a measure on $(X, 2^X)$, and for every extended real valued non-negative measurable function f on X,

$$\int_X f d\mu = \sum_{i=1}^{n} f(x_i) w_i.$$

♢

Exercise 3.22. Suppose $X = \{x_1, x_2, \ldots\}$, and $w_1, w_2, \ldots$ are non-negative reals. For $x \in X$, define $w(x) = w_j$ whenever $x = x_j$, and $\mu(E) = \sum_{x \in E} w(x)$ for $E \subseteq X$. Show that μ is a measure on $(X, 2^X)$, and for every extended real valued non-negative measurable function f on X,

$$\int_X f d\mu = \sum_{i=1}^{n} f(x_i) w_i.$$

♢

3.4 Integral of Complex Measurable Functions

Recall that if α is a real number, then $\alpha = \alpha^+ - \alpha^-$ and $|\alpha| = \alpha^+ + \alpha^-$, where $\alpha^+ := \max\{\alpha, 0\}$ and $\alpha^- := \max\{-\alpha, 0\}$. Similarly, if f is a real valued function defined on a set X, then

$$f = f^+ - f^-, \qquad |f| = f^+ + f^-,$$

where f^+ and f^- are defined as

$$f^+ := \max\{f, 0\}, \qquad f^- := \max\{-f, 0\}.$$

Now, let $(X, \mathcal{A}, \mu)$ be a measure space and f be a real measurable function on X. Then we know that both f^+ and f^- are positive measurable functions. Hence, we can define $\int_X f^+ d\mu$ and $\int_X f^- d\mu$.

In view of this we have the following definition.

Definition 3.23. Suppose f is a real measurable function on X. If at least one of $\int_X f^+ d\mu$ and $\int_X f^- d\mu$ is finite, then we define

$$\int_X f d\mu = \int_X f^+ d\mu - \int_X f^- d\mu.$$

$\diamondsuit$

Since $|f| = f^+ + f^-$, it follows that if $\int_X |f| d\mu < \infty$, then both $\int_X f^+ d\mu$ and $\int_X f^- d\mu$ are finite, and in that case $\int_X f d\mu = \int_X f^+ d\mu - \int_X f^- d\mu$. This observation prompts the following definition.

Definition 3.24. A real measurable function f on X is said to be **integrable** (over X) if $\displaystyle\int_X |f| d\mu < \infty$. $\diamondsuit$

Nothing prevents us from having the following definition as well.

Definition 3.25. A complex valued measurable function f on X is said to be **integrable** (over X) if $\displaystyle\int_X |f| d\mu < \infty$. $\diamondsuit$

We denote the set of all complex valued integrable functions by $\mathcal{L}(X, \mathcal{A}, \mu)$ or simply by $\mathcal{L}(\mu)$ if the set X and the sigma algebra are understood.

We may recall that if f is any complex valued function on X, then $f = \mathrm{Re} f + i \mathrm{Im} f$.

We may observe that if $f \in \mathcal{L}(\mu)$, then $\mathrm{Re} f$ and $\mathrm{Im} f$ are integrable. In view of this we have the following definition.

Definition 3.26. If $f \in \mathcal{L}(\mu)$, then we define the integral of f (over X) with respect to μ as

$$\int_X f \, d\mu := \int_X \mathrm{Re} f \, d\mu + i \int_X \mathrm{Im} f \, d\mu.$$

$\diamondsuit$

Notation. We shall denote the integral $\displaystyle\int_X f \, d\mu$ by $\displaystyle\int_X f$ or by $\displaystyle\int f$.

Exercise 3.27. If $f \in \mathcal{L}(\mu)$ such that $\int_X f \geq 0$, then show that

$$\int_X f = \int_X \mathrm{Re} f \leq \int_X |f|.$$

$\diamondsuit$

THEOREM 3.28. *If f and g are in $\mathcal{L}(\mu)$ and $\alpha \in \mathbb{C}$, then $f+g$ and αf are in $\mathcal{L}(\mu)$, and*

$$\int_X (f+g) = \int_X f + \int_X g, \qquad \int_X \alpha f = \alpha \int_X f.$$

Proof. Suppose f and g are in $\mathcal{L}(\mu)$ and $\alpha \in \mathbb{C}$. Since $|f+g| \leq |f|+|g|$ and $|\alpha f| = |\alpha|\,|f|$ it follows that $f+g$ and αf are in $\mathcal{L}(\mu)$.

Now to prove the equalities of the integrals, we consider two cases.

Case 1: f and g are real valued.

In this case, we have

$$(f+g)^+ - (f+g)^- = f+g = (f^+ - f^-) + (g^+ - g^-).$$

Thus,

$$(f+g)^+ + f^- + g^- = (f+g)^- + f^+ + g^+.$$

Hence,

$$\int_X (f+g)^+ + \int_X f^- + \int_X g^- = \int_X (f+g)^- + \int_X f^+ + \int_X g^+$$

so that

$$\begin{aligned} \int_X (f+g) &= \int_X (f+g)^+ - \int_X (f+g)^- \\ &= \int_X f^+ - \int_X f^- + \int_X g^+ - \int_X g^- \\ &= \int_X f + \int_X g. \end{aligned}$$

Case 2: f and g are complex valued.

Let $h := f+g$, and let $f_1 = \mathrm{Re} f$, $g_1 = \mathrm{Re} g$, $h_1 = \mathrm{Re} h$, $f_2 = \mathrm{Im} f$, $g_2 = \mathrm{Im} g$, $h_2 = \mathrm{Im} h$. Then

$$h = h_1 + i h_2 = (f_1 + g_1) + i(f_2 + g_2)$$

so that

$$\int_X h = \int_X (f_1 + g_1) + i \int_X (f_2 + g_2).$$

By case 1,

$$\int_X (f_1 + g_1) = \int_X f_1 + \int_X g_1, \qquad \int_X (f_2 + g_2) = \int_X f_2 + \int_X g_2.$$

Hence,

$$\begin{aligned} \int_X (f+g) &= \left(\int_X f_1 + \int_X g_1\right) + i\left(\int_X f_2 + \int_X g_2\right) \\ &= \left(\int_X f_1 + i\int_X f_2\right) + \left(\int_X g_1 + i\int_X g_2\right) \\ &= \int_X f + \int_X g. \end{aligned}$$

Similarly we can reduce the case of complex valued f and complex number α to the case of real valued f and real number α, and then prove $\int_X \alpha f = \alpha \int_X f$. □

Exercise 3.29. Show that $\mathcal{L}(\mu)$ is a vector space over $\mathbb{C}$, and $f \mapsto \int_X f$ is a linear functional on $\mathcal{L}(\mu)$. $\diamondsuit$

THEOREM 3.30. *If $f \in \mathcal{L}(\mu)$, then*

$$\left|\int_X f\right| \leq \int_X |f|.$$

Proof. Let $f \in \mathcal{L}(\mu)$, and let $\theta \in \mathbb{R}$ be such that $\int_X f = \left|\int_X f\right| e^{i\theta}$. Then (cf. Exercise 3.27)

$$\left|\int_X f\right| = e^{-i\theta} \int_X f = \int_X e^{-i\theta} f = \int_X \mathrm{Re}(e^{-i\theta} f) \leq \int_X |f|.$$

This completes the proof. $\square$

Exercise 3.31. Show that $f \mapsto \int_X |f|$ is a semi-norm on the vector space $\mathcal{L}(\mu)$. $\diamondsuit$

Exercise 3.32. Show that $\mathcal{N} := \{f \in \mathcal{L}(\mu) : \int_X |f| = 0\}$ is subspace of $\mathcal{L}(\mu)$, and $[f] \mapsto \int_X |f|$ is a norm on the quotient space $\mathcal{L}(\mu)/\mathcal{N}$. $\diamondsuit$

CONVENTION. The quotient space $\mathcal{L}(\mu)/\mathcal{N}$ considered in Exercise 3.32 is denoted by $L(\mu)$. Abusing the notation, we shall denote the element $[f]$ in $L(\mu)$ by f.

Exercise 3.33. Prove that, if $f \in \mathcal{L}(\mu)$, then $E \in \mathcal{A}$, $\mu(E) = 0$ imply $\int_E f\, d\mu = 0$. $\diamondsuit$

In view of the above exercise, a set of measure zero has no effect as far as integral of a function is concerned.

Definition 3.34. Let $(X, \mathcal{A}, \mu)$ be a measure space and $E \in \mathcal{A}$. We say that a property P concerning points in X holds **almost everywhere** on E, or P holds for **almost all** $x \in E$, if the set $\{x \in E : P \text{ does not hold at } x\}$ is measurable and is of measure zero, i.e.,

$$\mu\left(\{x \in E : P(x) \text{ does not hold at } x\}\right) = 0.$$

In this case, we write P holds a.e. on E. $\diamondsuit$

Using the above definition, we can state that if f and g are in $\mathcal{L}(\mu)$ such that $f = g$ a.e. on X, then $\int_X f = \int_X g$.

PROPOSITION 3.35. *Suppose $f : X \to [0, \infty]$ and $E \in \mathcal{A}$. If $\int_E f = 0$, then $f = 0$ a.e. on E.*

Proof. Suppose $\displaystyle\int_E f = 0$. Note that $F := \{x \in E : f(x) \neq 0\} = \cup_{n=1}^{\infty} E_n$ where $E_n := \{x : f(x) > 1/n\}$. Hence

$$0 = \int_E f \geq \int_F f \geq \int_{E_n} f \geq \frac{1}{n}\mu(E_n).$$

Hence $\mu(E_n) = 0$ for all $n \in \mathbb{N}$ so that $\mu(F) = 0$. Thus, $f = 0$ a.e. on E. $\square$

Exercise 3.36. Prove that, if f and g are complex measurable functions such that $f = 0$ a.e. on X and $g = 0$ a.e. on X, then $f + g = 0$ a.e. on X. ♢

Exercise 3.37. Prove that, if $f \in \mathcal{L}(\mu)$ is such that $\int_E f = 0$ for all $E \in \mathcal{A}$, then $f = 0$ a.e.

Hint: It is enough to prove for the case of real valued f. Take $E = \{x \in X : f(x) \geq 0\}$ and show that $\int_X f^+ = 0$. Similarly show that $\int_X f^- = 0$. ♢

THEOREM 3.38. (Dominated convergence theorem) *Suppose* (f_n) *is a sequence of complex measurable functions such that* $(f_n(x))$ *converges for every* $x \in X$*, and let* $f(x) := \lim_{n\to\infty} f_n(x)$*,* $x \in X$*. Suppose there exists* $g \in \mathcal{L}(\mu)$ *such that* $|f_n| \leq |g|$ *for all* $n \in \mathbb{N}$*. Then* f_n *and* f *are integrable, and*

$$\lim_{n\to\infty} \int_X |f_n - f| = 0.$$

In particular,

$$\lim_{n\to\infty} \int_X f_n = \int_X f.$$

Proof. Since $|f_n| \leq |g|$ for some $g \in \mathcal{L}(\mu)$ and $(f_n(x))$ converges for every $x \in X$, it follows that $|f| \leq |g|$ and $f_n \in \mathcal{L}(\mu)$, $f \in \mathcal{L}(\mu)$. Also, we note that $2|g| - |f_n - f| \geq 0$ for all $n \in \mathbb{N}$, and $2|g| - |f_n - f| \to 2|g|$ as $n \to \infty$. Hence, by Fatou's lemma,

$$\begin{aligned}\int_X 2|g| &= \int_X \liminf_n (2|g| - |f_n - f|) \\ &\leq \liminf_n \int_X (2|g| - |f_n - f|) \\ &= \int_X 2|g| - \limsup_n \int_X |f_n - f|.\end{aligned}$$

Thus,

$$0 \leq \liminf_n \int_X |f_n - f| \leq \limsup_n \int_X |f_n - f| \leq 0,$$

consequently, $\lim_{n\to\infty} \int_X |f_n - f|$ exists and it is equal to 0. □

Definition 3.39. Let $(X, \mathcal{A}, \mu)$ be a measure space, $E \in \mathcal{A}$ and (f_n) is a sequence of complex valued measurable functions defined on E. Then we say (f_n) converges to f almost everywhere on E, denoted by $f_n \to f$ a.e. on E, if and only if $\mu(\{x \in E : f_n(x) \not\to f(x)\}) = 0$. ♢

Remark 3.40. The conclusions in Theorem 3.38 holds good if we replace *pointwise convergence* by *convergence almost everywhere.* ♢

LEMMA 3.41. *Suppose* $f : X \to [0, \infty]$ *is such that* $\int_X f < \infty$*. Then* $f(x) \in \mathbb{R}$ *for almost all* $x \in X$*, i.e.,* $\mu(\{x : f(x) = \infty\}) = 0$*.*

Proof. We note that $F := \{x : f(x) = \infty\} = \cap_{n=1}^{\infty} A_n$, where $A_n := \{x : f(x) > n\}$. Then $F \subseteq A_n$ for all $n \in \mathbb{N}$ and we have $\mu(F) \leq \mu(A_n)$ for all $n \in \mathbb{N}$. Thus,

$$\int_X f \geq \int_{A_n} f = \int_X f\chi_{A_n} \geq \int_X n\chi_{A_n} = n\mu(A_n) \geq n\mu(F).$$

Thus

$$\mu(F) \leq \frac{1}{n} \int_X f \quad \forall\, n \in \mathbb{N},$$

so that $\mu(F) = 0$. □

THEOREM 3.42. *Suppose (f_n) is a sequence of complex measurable functions such that $\sum_{n=1}^{\infty} \int_X |f_n|$ converges. Then $\sum_{n=1}^{\infty} f_n$ converges for almost all $x \in X$, and*

$$\int_X \left(\sum_{n=1}^{\infty} f_n \right) = \sum_{n=1}^{\infty} \int_X f_n.$$

Proof. Let $g_n := \sum_{j=1}^{n} f_j$. Then $|g_n| \leq \sum_{j=1}^{n} |f_j(x)| \leq \sum_{j=1}^{\infty} |f_j(x)|$. Let $h := \sum_{n=1}^{\infty} |f_n|$. Since

$$\int_X h = \sum_{n=1}^{\infty} \int_X |f_n| < \infty,$$

by Lemma 3.41, $h = \sum_{n=1}^{\infty} |f_n|$ is real valued almost everywhere. Let $S := \{x \in X : h(x) < \infty\}$. Then we have $\mu(S^c) = 0$, and $g(x) := \sum_{n=1}^{\infty} f_n(x)$ converges for every $x \in S$. Then we have $|g_n| \leq h$ on S and $h \in \mathcal{L}(\mu)$, and $g_n(x) \to g(x)$ for every $x \in S$. Hence, by Dominated convergence Theorem, $g \in \mathcal{L}(\mu)$ and $\int_S g_n \to \int_S g$ as $n \to \infty$. But,

$$\int_S g_n = \int_X \left(\sum_{j=1}^{n} f_j \right) = \sum_{j=1}^{n} \int_X f_j, \qquad \int_S g = \int_X g = \int_X \left(\sum_{n=1}^{\infty} f_n \right).$$

Thus,

$$\sum_{j=1}^{\infty} \int_X f_j = \lim_{n\to\infty} \sum_{j=1}^{n} \int_X f_j = \int_X \left(\sum_{n=1}^{\infty} f_n \right).$$

Thus the proof is completed. □

4 Lecture 4: L^p-spaces

For $1 \le p < \infty$ and for a complex valued measurable function f, let

$$\|f\|_p := \left(\int_X |f|^p d\mu\right)^{1/p}.$$

THEOREM 4.1. *(***Hölder's inequality***)* Let $(X, \mathcal{A}, \mu)$ be a measure space and f, g be complex measurable functions on X. For $1 < p < \infty$, let $q > 1$ be such that $p + q = pq$. Then

$$\int_X |fg| d\mu \le \|f\|_p \, \|g\|_q.$$

Proof. First we observe that if one of $\|f\|_p$ and $\|g\|_q$ is zero, then the inequality holds. Hence, assume that both $\|f\|_p$ and $\|g\|_q$ are strictly positive. Also, if one of $\|f\|_p$ and $\|g\|_q$ is infinity, then the inequality holds. Thus, it is enough to prove by assuming that $0 < \|f\|_p < \infty$ and $0 < \|g\|_q < \infty$. For this purpose, first we may observe that for any non-negative real numbers a and b,

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}.$$

Thus, for $x \in X$, taking $a = |f(x)|/\|f\|_p$ and $b = |g(x)|/\|g\|_q$, we have

$$\frac{|f(x)g(x)|}{\|f\|_p \|g\|_q} \le \frac{|f(x)|^p}{p\|f\|_p^p} + \frac{|g(x)|^q}{q\|g\|_q^q}.$$

Now, taking integrals over X, we get

$$\frac{1}{\|f\|_p \|g\|_q} \int_X |fg| \le \frac{\int_X |f|^p}{p\|f\|_p^p} + \frac{\int_X |g|^q}{q\|g\|_q^q} = 1.$$

Hence, $\int_X |fg| \le \|f\|_p \|g\|_q$, which completes the proof. □

THEOREM 4.2. *(***Minkowski inequality***)* Let $(X, \mathcal{A}, \mu)$ be a measure space and f, g be complex measurable functions on X. Then for $1 \le p < \infty$,

$$\|f + g\|_p \le \|f\|_p + \|g\|_p.$$

Proof. First we observe that if one of $\|f\|_p$ and $\|g\|_p$ is infinity, then the inequality holds. Hence, it is enough to prove for the case when $\|f\|_p < \infty$ and $\|g\|_p < \infty$. We note that if $p = 1$, then the proof is trivial, as the result follows by observing $|f + g| \le |f| + |g|$. Thus assume that $1 < p < \infty$. Also, we can assume that $\|f + g\|_p \ne 0$. Note that

$$\int_X |f + g|^p d\mu = \int_X |f + g|^{p-1}|f + g| d\mu \le \int_X |f + g|^{p-1}|f| d\mu + \int_X |f + g|^{p-1}|g| d\mu.$$

Hence, with $q > 1$ satisfying $p + q = pq$ and applying Hölder's inequality, we have

$$\int_X |f + g|^{p-1}|f| \le \|f\|_p \left(\int_X |f + g|^{(p-1)q}\right)^{1/q} = \|f\|_p \|f + g\|_p^{q/q},$$

$$\int_X |f+g|^{p-1}|f| \leq \|g\|_p \left(\int_X |f+g|^{(p-1)q}\right)^{1/q} = \|f\|_p \|f+g\|_p^{q/q}.$$

Thus,

$$\|f+g\|_p^p = \int_X |f+g|^p d\mu \leq (\|f\|_p + \|g\|_p) \|f+g\|_p^{q/q}.$$

Now canceling out $\|f+g\|_p^{q/q}$, as $\|f+g\|_p \neq 0$, we get $\|f+g\|_p \leq \|f\|_p + \|g\|_q$. □

For $1 \leq p < \infty$, let

$$\mathcal{L}^p := \mathcal{L}^p(X, \mathcal{A}, \mu) := \{f : X \to \mathbb{C} : f \text{ measrable and } \|f\|_p < \infty\}.$$

and

$$\mathcal{Z}_p := \{f \in \mathcal{L}^p : f = 0 \text{ a.e. }\}.$$

Definition 4.3. A measurable function $f : X \to \mathbb{C}$ is said to be an **essentially bounded** function if there exists $M_f > 0$ such that $|f| \leq M_f$ a.e. on X. ♢

The set of all essentially bounded functions is denoted by $\mathcal{L}^\infty := \mathcal{L}^\infty(X, \mathcal{A}, \mu)$, and we denote

$$\|f\|_\infty := \inf\{M_f > 0 : |f| \leq M_f \text{ a.e.}\},$$

and

$$\mathcal{Z}_\infty := \{f \in \mathcal{L}^\infty : f = 0 \text{ a.e.}\}.$$

With the above definitions and the Minkowski's inequality, it follows that for any p satisfying $1 \leq p \leq \infty$,

$$\|f+g\|_p \leq \|f\|_p + \|g\|_q \quad \forall\, f, g \in \mathcal{L}^p(\mu).$$

Now, the proof of the following theorem is immediate.

THEOREM 4.4. *For $1 \leq p \leq \infty$, $\mathcal{L}^p$ is a complex vector space, and the map $f \mapsto \|f\|_p$ is a seminorm on $\mathcal{L}^p$. Also, the map $[f] \mapsto \|f\|_p$ is a norm on $\mathcal{L}^p/\mathcal{Z}_p$.*

CONVENTION. The quotient space $\mathcal{L}^p(\mu)/\mathcal{Z}_p$ considered in the above theorem is denoted by $L^p(X, \mathcal{A}, \mu)$ or simply by $L^p(\mu)$ or $L^p(X)$ or by L^p. Abusing the notation, we shall denote the element $[f]$ in $L(\mu)$ by f.

Observe that, if $X = \mathbb{N}$, $\mathcal{A} = 2^{\mathbb{N}}$ and μ is the counting measure, then for any $p \in [1, \infty]$, $\mathcal{Z}_p = \{0\}$ and

$$L^p = \mathcal{L}^p = \ell^p(\mathbb{N}).$$

Also, if $X = \{1, \ldots, k\}$ for some $k \in \mathbb{N}$, $\mathcal{A} = 2^{\mathbb{N}}$ and μ is the counting measure, then for any $p \in [1, \infty]$, $\mathcal{Z}_p = \{0\}$ and

$$L^p = \mathcal{L}^p = \mathbb{C}^k.$$

Exercise 4.5. Show that if $\mu(X) < \infty$, then for $1 \le p \le r \le \infty$,

$$L^\infty \subseteq L^r \subseteq L^p \subseteq L^1.$$

$\diamondsuit$

Exercise 4.6. Show that if for $1 \le p \le r \le \infty$,

$$\ell^\infty(\mathbb{N}) \supseteq \ell^r(\mathbb{N}) \supseteq \ell^p(\mathbb{N}) \supseteq \ell(\mathbb{N})^1.$$

$\diamondsuit$

Next we prove the completeness of $L^p(\mu)$. Before that, let us observe the following.

LEMMA 4.7. *Let $1 \le p < \infty$ and let (f_n) be a Cauchy sequence in $L^p(\mu)$ having a subsequence which converges almost everywhere to a function f. Then $f \in L^p(\mu)$ and (f_n) converges to f in $L^p(\mu)$.*

Proof. Suppose (f_n) is a Cauchy sequence in $L^p(\mu)$ with $1 \le p < \infty$ having a subsequence (f_{n_k}) which converges a.e. to a function f. Then for each $n \in \mathbb{N}$, $|f_{n_k} - f_n| \to |f - f_n|$ a.e. on X. Therefore, by Fatou's lemma,

$$\int_X |f - f_n|^p \le \liminf_k \int_X |f_{n_k} - f_n|^p.$$

Now, let $\varepsilon > 0$ and let $N \in \mathbb{N}$ be such that $\|f_k - f_n\|_p < \varepsilon$ for all $n, k \ge N$. Then $\|f_{n_k} - f_n\|_p < \varepsilon$ for all $n, k \ge N$ so that

$$\int_X |f - f_n|^p \le \liminf_n \int_X |f_{n_k} - f_n|^p < \varepsilon^p \quad \forall\, n \ge N,$$

showing that $f \in L^p$ and $\|f - f_n\|_p \to 0$ as $n \to \infty$. $\square$

We shall also make use of the following result.

THEOREM 4.8. *Let Ω be any nonempty set. Then $\ell^\infty(\Omega)$, the space of all bounded complex valued functions on Ω, is a Banach space with respect to the supremum norm,*

$$\|f\|_\infty := \sup_{t\in\Omega} |f(t)|, \quad f \in \ell^\infty(\Omega).$$

Proof. Let (f_n) be a Cauchy sequence in $\ell^\infty(\Omega)$. Then for every $t \in \Omega$, the sequence $(f_n(t))$ is a Cauchy sequence of complex numbers. Hence, $(f_n(t))$ converges for every $t \in \Omega$. Let

$$f(t) := \lim_{n\to\infty} f_n(t), \quad t \in \Omega.$$

Now, let $k \in \mathbb{N}$ such that $\|f_n - f_k\|_\infty \le 1$ for all $n \ge k$. Then we have

$$|f_n(t)| \le \|f_n - f_k\|_\infty + \|f_k\|_\infty < 1 + \|f_k\|_\infty$$

for all $n \ge N$ and for all $t \in \Omega$. Hence,

$$|f(t)| \le \max\{1 + \|f_k\|_\infty, \|f_k\|_1, \|f_k\|_2, \ldots, \|f_k\|_k\} \quad \forall\, t \in \Omega$$

showing that $f \in \ell^\infty(\Omega)$. Next we show that $\|f - f_n\|_\infty \to 0$ as $n \to \infty$. For this let $\varepsilon > 0$ be given. Let $N \in \mathbb{N}$ be such that $\|f_n - f_m\|_\infty \leq \varepsilon$ for all $n, m \geq N$. Let $t \, in\Omega$ and let $m \geq N$ be such that $\|f(t) - f_m(t)| < \varepsilon$. Then we have

$$|f(t) - f_n(t)| \leq |f(t) - f_m(t)| + |f_m(t) - f_n(t)| < 2\varepsilon$$

for all $n \geq N$. Note that this N is independent of x. Hence we have proved that $\|f - f_n\|_\infty < 2\varepsilon$ for all $n \geq N$. Thus, the Cauchy sequence (f_n) converges in $\ell^\infty(\Omega)$. □

THEOREM 4.9. *For $1 \leq p \leq \infty$, L^p is a Banach space with respect to the norm $f \mapsto \|f\|_p$.*

Proof. First we consider the case $1 \leq p < \infty$: Let (f_n) be a Cauchy sequence in L^p. By Lemma 4.7, it is enough to show that (f_n) has a subsequence which converges almost every where.

Since (f_n) is a Cauchy sequence, for each $i \in \mathbb{N}$, there exists $n_i \in \mathbb{N}$ such that $\|f_n - f_{n_i}\|_p < 1/2^i$ for all $n \geq n_i$. Without loss of generality we may assume that $n_i \leq n_{i+1}$ for all $i \in \mathbb{N}$. Then we have $\|f_{n_{i+1}} - f_{n_i}\|_p < 1/2^i$ for all $i \in \mathbb{N}$. Note that

$$f_{n_k} = \sum_{i=1}^{k-1}(f_{n_{i+1}} - f_{n_i}).$$

Thus, it is enough to show that $\sum_{i=1}^{\infty}(f_{n_{i+1}} - f_{n_i})$ converges a.e. on X. Now, let

$$g_k := \sum_{i=1}^{k} |f_{n_{i+1}} - f_{n_i}|, \qquad g := \sum_{i=1}^{\infty} |f_{n_{i+1}} - f_{n_i}|.$$

Then $g_k(x) \to g(x)$ as $k \to \infty$ for every $x \in X$, and by Monotone Convergence Theorem, $\lim_{k\to\infty} \int_X g_k^p = \int_X g^p$. But,

$$\left(\int_X g_k^p\right)^{1/p} = \|g_k\|_p \leq \sum_{i=1}^{k} \|f_{n_i} - f_{n_{i-1}}\|_p \leq 1.$$

Hence, $\int_X g^p < \infty$. Therefore, by Lemma 3.41, $g(x) < \infty$ a.e. on X. Thus $\sum_{i=1}^{\infty}(f_{n_{i+1}} - f_{n_i})$ converges a.e. on X, completing the proof for the case $1 \leq p < \infty$.

Next we consider the case $p = \infty$. Let (f_n) be a Cauchy sequence in L^∞. Then the sets

$$A_k := \{x \in X : |f_k(x)| > \|f_k\|_\infty\},$$

$$B_{m,n} := \{x \in X : |f_n(x) - f_m(x)| > \|f_n - f_m\|_\infty\}$$

are of measure zero for every $k, m, n \in \mathbb{N}$. Hence, the set $E := (\cup_k A_k) \cup (\cup_{m,n} B_{m,n})$ is of measure zero, and for each $x \in \Omega := E^c$,

$$|f_k(x)| \leq \|f_k\|_\infty \, \forall \, k \in \mathbb{N}, \qquad |f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty \, \forall \, m, n \in \mathbb{N}.$$

Since (f_n) is cauchy in L^∞, it follows that $(\tilde{f}_n)$ with $\tilde{f}_n = f_n|_\Omega$ is a Cauchy sequence in $\ell^\infty(\Omega)$. Since $\ell^\infty(\Omega)$ is a Banach space, it follows that there exists $\tilde{f} \in \ell^\infty(\Omega)$ such that $\|\tilde{f}_n - \tilde{f}\|_{\ell^\infty(\Omega)} \to 0$ as $n \to \infty$. Defining $f(x) = \tilde{f}(x)$ for $x \in \Omega$ and $f(x) = 0$ for $x \notin \Omega$, it follows that $f \in L^\infty$, and $\|\tilde{f}_n - \tilde{f}\|_\infty \to 0$ as $n \to \infty$. $\square$

Note that as part of the proof of the above theorem, we have proved the following.

PROPOSITION 4.10. *If $1 \le p < \infty$, then every Cauchy sequence in $L^p(\mu)$ has a subsequence which converges almost everywhere.*

In particular, if (f_n) is a sequence in $L^p(\mu)$ which converges to $f \in L^p(\mu)$, then (f_n) has a subsequence which converges a.e. to $f \in L^p(\mu)$.

4.1 Sobolev spaces

One of the most important measure spaces which is important in applications is $(\Omega, \mathcal{A}, \mu)$, where Ω is an open subset of $\mathbb{R}^k$, $\mathcal{A}$ is the σ-algebra of all Borel subsets or Lebesgue measurable subsets of Ω and μ is the Lebesgue measure. For further discussion, let us introduce the following:

- $C(\Omega)$: The set of all continuous ($\mathbb{C}$-valued) functions on Ω.
- For $\alpha := (\alpha_1, \ldots, \alpha_k)$ with $\alpha_j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$,
$$|\alpha| = \alpha_1 + \ldots + \alpha_k, \qquad D^\alpha := \Big(\frac{\partial}{\partial x_1}\Big)^{\alpha_1} \Big(\frac{\partial}{\partial x_2}\Big)^{\alpha_2} \cdots \Big(\frac{\partial}{\partial x_k}\Big)^{\alpha_k}.$$
- For $m \in \mathbb{N}$, $C^m(\Omega)$: The set of all $f \in C(\Omega)$ such that $D^\alpha f$ exists and $D^\alpha f \in C(\Omega)$ for all α with $|\alpha| \le m$.
- $C^\infty(\Omega) = \bigcap_{m=1}^{\infty} C^m(\Omega)$.
- For $f : \Omega \to \mathbb{C}$, the support of f is the set
$$\operatorname{supp}(f) := \operatorname{cl}\{x \in \Omega : f(x) \ne 0\}.$$
- A function A $f : \Omega \to \mathbb{C}$ is said to be of **compact support** if $\operatorname{supp}(f)$ is compact.
- $C_c(\Omega)$: The set of all functions in $C(\Omega)$ with compact support.
- For $m \in \mathbb{N}$, $C_c^m(\Omega) := C^m(\Omega) \cap C_c(\Omega)$.
- $C_c^\infty(\Omega) := C^\infty(\Omega) \cap C_c(\Omega)$.

We observe that
$$C^\infty(\Omega) \subseteq C^m(\Omega) \subseteq C(\Omega) \quad \forall\, m \in \mathbb{N},$$
$$C_c^\infty(\Omega) \subseteq C_c^m(\Omega) \subseteq C_c(\Omega) \subseteq L^p(\Omega) \quad \forall\, m \in \mathbb{N}.$$
For $m \in \mathbb{N}_0$ and $f \in C^m(\Omega)$, let
$$\|f\|_{m,p} = \Big(\sum_{|\alpha| \le m} \|D^\alpha f\|_p^p\Big)^{1/p}, \qquad 1 \le p < \infty,$$
$$\|f\|_{m,\infty} = \max\{\|D^\alpha f\|_\infty : |\alpha| \le m\}.$$

THEOREM 4.11. *For $m \in \mathbb{N}_0$ and $1 \le p \le \infty$,*

$$\mathcal{W}^{m,p}(\Omega) := \{f \in C^m(\Omega) : \|f\|_{m,p} < \infty\}$$

is a linear space and

$$f \mapsto \|f\|_{m,p}$$

is a norm on $\mathcal{W}^{m,p}(\Omega)$. Further,

$$\langle f, g\rangle := \sum_{|\alpha|\le m} \langle D^\alpha f, D^\alpha g\rangle_{L^2(\Omega)} = \sum_{|\alpha|\le m} \int_\Omega D^\alpha f \overline{D^\alpha g} d\mu$$

defines an inner product on $\mathcal{W}^{m,2}(\Omega)$.

Definition 4.12. The completion of $\mathcal{W}^{m,p}(\Omega)$ is called a **Sobolev space**, and it is denoted by $W^{m,p}(\Omega)$. The Hilbert space $W^{m,2}(\Omega)$ is usually denoted by $H^m(\Omega)$. ♢

We note that $C_c^m(\Omega)$ is subspace of $W^{m,p}(\Omega)$.

- The closure of $C_c^m(\Omega)$ in $W^{m,p}(\Omega)$ is denoted by $W_0^{m,p}(\Omega)$, and the closure of $C_c^m(\Omega)$ in $H^m(\Omega)$ is denoted by $H_0^m(\Omega)$.

Definition 4.13. Let $f \in L^p(\Omega)$ for $1 \le p < \infty$. Then f is said to have **generalized derivative of order** $\alpha \in \mathbb{N}_0^k$ if there exists $f_\alpha \in L^1_{loc}(\Omega)$ such that

$$\langle f_\alpha, \varphi\rangle_{L^2(\Omega)} = (-1)^{|\alpha|}\langle f, D^\alpha \varphi\rangle \quad \forall\, \varphi \in C_c^\infty(\Omega),$$

and in that case f_α is called the **generalized derivative of f of order** α, and we write this f_α by $f^{(\alpha)}$. ♢

It is known that

$$W^{m,p}(\Omega) = \{f \in L^p(\Omega) : \ f^{(\alpha)} \text{ exists for } |\alpha| \le m \text{ and } f^{(\alpha)} \in L^p(\Omega)\}$$

and

$$\|f\|_{m,p} = \Big(\sum_{|\alpha|\le m} \|f^{(\alpha)}\|_p^p\Big)^{1/p}.$$

5 Problems

1. Let $E \subseteq \mathbb{R}$. Prove that for every $\varepsilon > 0$ there exists a countable family $\{I_n\}$ of open intervals such that $E \subseteq \bigcup_n^{\infty} I_n$ and $\sum_n \ell(I_n) \leq m^*(E) + \varepsilon$. Strict inequality occurs in the above if $m^*(E) < \infty$.

2. Prove the following:

   (i) $m^*(\varnothing) = 0$.

   (ii) If E is a singleton set, then $m^*(E) = 0$.

   (iii) If E is a countable set, then $m^*(E) = 0$.

   (iv) If $E = (a, b)$ with $a < b$ then $m^*(E) \leq b - a$.

3. Prove the following:

   (i) $E_1 \subseteq E_2 \Rightarrow m^*(E_1) \leq m^*(E_2)$.

   (ii) $m^*(A \cup B) \leq m^*(A) + m^*(B)$.

   (iii) If $E \subseteq A$ and $m^*(E) = 0$, then $m^*(A \setminus E) = m^*(A)$.

   (iv) If $E \subseteq \mathbb{R}$ and $x \in \mathbb{R}$, then $m^*(E + x) = m^*(E)$.

4. Prove that for every sequence (I_n) of open intervals, $m^*(\bigcup I_n) \leq \sum_n \ell(I_n)$.

5. If $E \subseteq \mathbb{R}$, then for every $\varepsilon > 0$, there exists an open set $G \subseteq \mathbb{R}$ such that $G \supseteq E$ and $m^*(G) \leq m^*(E) + \varepsilon$.

6. Derive Problem 1.4(iii) from Problem1.4(ii) by using sub-additivity of m^*.

7. Prove that if
$$m^*(A_1 \cup A_2) = m^*(A_1) + m^*(A_2). \tag{5.3}$$
holds for any two disjoint sets A_1 and A_2, then
$$m^*\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} m^*(A_n) \tag{5.4}$$
holds for any pairwise disjoint denumerable family $\{A_n\}_{n=1}^{\infty}$.

   Prove that (5.4) need not hold for every denumerable disjoint family $\{A_n\}_{n=1}^{\infty}$.

   Hint: On $[0, 1]$ consider the relation $x \sim y \iff x - y \in \mathbb{Q}$.

   • Show that $\sim$ is an equivalence relation on $[0, 1]$.

   Let E be the subset of $[0, 1]$ such that its intersection with each equivalence class is a singleton set. (Such a set E exists by using the *axiom of choice*.) Let $\{r_1, r_2, \ldots\} = \mathbb{Q} \cap [-1, 1]$ and $E_n := E + r_n$ for $n \in \mathbb{N}$.

   • Show that $\{E_n\}$ is a pairwise disjoint family.

   Observe: $[0, 1] \subseteq \bigcup_{n=1}^{\infty} E_n \subseteq [-1, 2]$ so that $1 \leq m^*\Big(\bigcup_{n=1}^{\infty} E_n\Big) \leq 3$. Therefore, if (1.2) is true, we arrive at a contradiction.

8. There exist disjoint subsets A_1 and A_2 of $\mathbb{R}$ such that

$$m^*(A_1 \cup A_2) \neq m^*(A_1) + m^*(A_2).$$

Why?

9. Prove the following:

(i) $E \in \mathcal{M} \iff m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c) \quad \forall\, A \subseteq \mathbb{R}$.

(ii) $\varnothing \in \mathcal{M}$.

(iii) $E \in \mathcal{M} \Rightarrow E^c \in \mathcal{M}$.

(iv) $m^*(E) = 0 \Rightarrow E \in \mathcal{M}$.

10. Prove that if $\{A_1, \ldots, A_n\}$ is a disjoint family in $\mathcal{M}$, then

$$m^*\Big(\bigcup_{i=1}^{n} A_i\Big) = \sum_{i=1}^{n} m^*(A_i).$$

[*Hint*: Use Theorem 1.18.]

11. Prove that if $\{A_n\}_{n=1}^{\infty}$ is a disjoint family in $\mathcal{M}$, then

$$m^*\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} m^*(A_n).$$

[*Hint*: Use Problem 10 and Problem 7.]

12. There exists $E \subseteq \mathbb{R}$ such that $E \notin \mathcal{M}$ - Why?.

13. If $E \in \mathcal{M}$ with $m(E) < \infty$ then show that for every $\varepsilon > 0$, there exists an open set $G \subseteq \mathbb{R}$ such that $G \supseteq E$ and $m(G \setminus E) < \varepsilon$.

14. Let $(X, \mathcal{A}, \mu)$ be a measure space and $E \in \mathcal{A}$. Prove that $\mathcal{A}_E := \{A \subset E : A \in \mathcal{A}\}$ is a σ-algebra on E and $\mu_E(A) := \mu(A)$, $A \in \mathcal{A}_E$, defines a measure on $\mathcal{A}_E$.

15. Let $\mu^* : 2^X \to [0, \infty]$ be an outer measure on X. Prove that

(i) $\mathcal{A} := \{E \subseteq X : \forall\, A \subseteq X,\, \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)\}$ is a σ-algebra on X, and

(ii) $\mu := \mu^*|_{\mathcal{A}}$ is a measure on $\mathcal{A}$.

16. Let $(X, \mathcal{A})$ be a measurable space and μ be a measure on it. For $A \subseteq X$, let

$$\mu^*(A) = \inf\{\sum_{i=1}^{\infty} \mu(E_i) : \{E_i\}_{i=1}^{\infty} \subseteq \mathcal{A} \text{ such that } A \subseteq \cup_{i=1}^{\infty} E_i\}.$$

Prove that μ^* is an outer measure and $\mu = \mu^*|_{\mathcal{A}}$.

17. Let $(X, \mathcal{A})$ be a measurable space. Show that a real valued function $f : X \to \mathbb{R}$ is measurable if and only if $f^{-1}(I) \in \mathcal{A}$ for every open interval I.

18. Let $(X, \mathcal{A})$ be a measurable space and $E \subseteq X$. Prove that χ_E is measurable if and only if $E \in \mathcal{A}$.

19. Let $(X, \mathcal{A})$ be a measurable space and φ is a simple function on X with canonical representation $\varphi = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$. Then φ is measurable if and only if $E_i \in \mathcal{A}$ for $i = 1, \ldots, n$ – Justify.

20. Suppose φ is a non-negative simple measurable function and E is a measurable set. Let μ_E be the restriction of μ to the restricted σ-algebra $\mathcal{A}_E := \{A \subseteq E : A \in \mathcal{A}\}$. Show that $\int_E \varphi d\mu = \int_E \varphi d\mu_E$.

21. Suppose $X = \{x_1, x_2, \ldots\}$ with the counting measure μ. If f is an extended real valued non-negative measurable function on X, then show that

$$\int_X f d\mu = \sum_{i=1}^{\infty} f(x_i).$$

22. Suppose $a_{ij} \geq 0$ for all $i, j \in \mathbb{N}$. Then show that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}.$$

23. Suppose $a_{ij} \in \mathbb{R}$ for all $i, j \in \mathbb{N}$ such that $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}| < \infty$. Show that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}.$$

24. Suppose $X = \{x_1, \ldots, x_n\}$, and $w_1, \ldots, w_n$ are non-negative reals. For $x \in X$, define $w(x) = w_j$ whenever $x = x_j$, and $\mu(E) = \sum_{x \in E} w(x)$ for $E \subseteq X$. Show that μ is a measure on $(X, 2^X)$, and for every extended real valued non-negative measurable function f on X,

$$\int_X f d\mu = \sum_{i=1}^{n} f(x_i) w_i.$$

25. Suppose $X = \{x_1, x_2, \ldots\}$, and $w_1, w_2, \ldots$ are non-negative reals. For $x \in X$, define $w(x) = w_j$ whenever $x = x_j$, and $\mu(E) = \sum_{x \in E} w(x)$ for $E \subseteq X$. Show that μ is a measure on $(X, 2^X)$, and for every extended real valued non-negative measurable function f on X,

$$\int_X f d\mu = \sum_{i=1}^{\infty} f(x_i) w_i.$$

26. If $f \in \mathcal{L}(\mu)$ such that $\int_X f \geq 0$, then show that

$$\int_X f = \int_X \mathrm{Re} f \leq \int_X |f|.$$

27. Show that $\mathcal{L}(\mu)$ is a vector space over $\mathbb{C}$, and $f \mapsto \int_X f$ is a linear functional.

28. Show that $f \mapsto \int_X |f|$ is a semi-norm on the vector space $\mathcal{L}(\mu)$.

29. Show that $\mathcal{N} := \{f \in \mathcal{L}(\mu) : \int_X |f| = 0\}$ is subspace of $\mathcal{L}(\mu)$, and $[f] \mapsto \int_X |f|$ is a norm on the quotient space $\mathcal{L}(\mu)/\mathcal{N}$.

30. Prove that, if f and g are complex measurable functions such that $f = 0$ a.e. on X and $g = 0$ a.e. on X, then $f + g = 0$ a.e. on X.

31. Prove that, if $f \in \mathcal{L}(\mu)$ is such that $\int_E f = 0$ for all $E \in \mathcal{A}$, then $f = 0$ a.e.

    Hint: It is enough to prove for the case of real valued f. Take $E = \{x \in X : f(x) \geq 0\}$ and show that $\int_X f^+ = 0$. Similarly show that $\int_X f^- = 0$.

32. Show that if $\mu(X) < \infty$, then for $1 \leq p \leq r \leq \infty$,

$$L^\infty \subseteq L^r \subseteq L^p \subseteq L^1.$$

33. Show that if for $1 \leq p \leq r \leq \infty$,

$$\ell^\infty(\mathbb{N}) \supseteq \ell^r(\mathbb{N}) \supseteq \ell^p(\mathbb{N}) \supseteq \ell^1(\mathbb{N}).$$

34. If $X = \mathbb{N}$, $\mathcal{A} = 2^{\mathbb{N}}$ and μ is the counting measure, then prove that $L^p = \mathcal{L}^p = \ell^p(\mathbb{N})$.

35. If $X = \{1, \ldots, k\}$ for some $k \in \mathbb{N}$, $\mathcal{A} = 2^{\mathbb{N}}$ and μ is the counting measure, then prove that $L^p = \mathcal{L}^p = \mathbb{C}^k$.