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ADJOINT OF UNBOUNDED OPERATORS ON BANACH SPACES

M.T. NAIR

Banach spaces considered below are over the field $\mathbb{K}$ which is either $\mathbb{R}$ or $\mathbb{C}$. Let $X$ be a Banach space. following Kato [2], $X^*$ denotes the linear space of all continuous conjugate linear functionals on $X$. We shall denote $\langle f, x \rangle := f(x), \ x \in X, \ f \in X^*.$

On $X^*$,

$$f \mapsto \|f\| := \sup_{\|x\|=1} |\langle f, x \rangle|$$

defines a norm on $X^*$.

**Definition 1.** The space $X^*$ is called the adjoint space of $X$.

Note that if $\mathbb{K} = \mathbb{R}$, then $X^*$ coincides with the dual space $X'$. It can be shown, analogues to the case of $X'$, that $X^*$ is a Banach space. Let $X$ and $Y$ be Banach spaces, and $A : D(A) \subseteq X \to Y$ be a densely defined linear operator. Now, we set out to define adjoint of $A$ as in Kato [2].

**Theorem 2.** There exists a linear operator $A^* : D(A^*) \subseteq Y^* \to X^*$ such that

$$\langle f, Ax \rangle = \langle A^* f, x \rangle \ \forall x \in D(A), \ f \in D(A^*)$$

and for any other linear operator $B : D(B) \subseteq Y^* \to X^*$ satisfying

$$\langle f, Ax \rangle = \langle Bf, x \rangle \ \forall x \in D(A), \ f \in D(B),$$

$D(B) \subseteq D(A^*)$ and $B$ is a restriction of $A^*$.

**Proof.** Suppose $D(A)$ is dense in $X$. Let

$$S := \{ f \in Y^* : x \mapsto \langle f, Ax \rangle \text{ continuous on } D(A) \}.$$ 

For $f \in S$, define $g_f : D(A) \to \mathbb{K}$ by

$$(g_f)(x) = \langle f, Ax \rangle \ \forall x \in D(A).$$

Since $D(A)$ is dense in $X$, $g_f$ has a unique continuous conjugate linear extension to all of $X$, preserving the norm. Let us denote this extension of $g_f$ by $\tilde{g}_f$. Taking $D(A^*) = S$, define $A^* : D(A^*) \to X^*$ by $A^* f = \tilde{g}_f$. It can be seen that $B$ is a linear operator, and it satisfies

$$\langle A^* f, x \rangle = \langle f, Ax \rangle \ \forall x \in D(A), \ f \in D(A^*).$$
Now, suppose \( B : D(B) \subseteq Y^* \to X^* \) is another linear operator such that
\[
\langle Bf, x \rangle = \langle f, Ax \rangle \quad \forall \ x \in D(A), \ f \in D(B).
\]

Note that if \( f \in D(B) \), then
\[
|\langle f, Ax \rangle| = |\langle Bf, x \rangle| \leq \|Bf\| \|x\| \quad \forall \ x \in D(A),
\]
so that \( x \mapsto \langle f, Ax \rangle \) is continuous on \( D(A) \). Thus, \( D(B) \subseteq S = D(A^*) \). Further, \( f \in D(B) \subseteq D(A^*) \) implies
\[
\langle Bf, x \rangle = \langle f, Ax \rangle = \langle A^* f, x \rangle \quad \forall \ x \in D(A).
\]

Hence, \( Bf = A^* f \) for all \( f \in D(B) \), showing that \( B \) is a restriction of \( A^* \). \( \square \)

**Definition 3.** Let \( A : D(A) \subseteq X \to Y \) be a densely defined linear operator. The operator \( A^* \) defined in Theorem 2 is called the adjoint of \( A \).

**Corollary 4.** If \( D(A) = X \) and \( A \) is a bounded operator, then \( A^* : Y^* \to X^* \) is the operator which satisfies
\[
(A^* f)(x) = f(Ax) \quad \forall \ f \in Y^*, \ x \in X,
\]
and \( A^* \) is a bounded linear operator with \( \|A^*\| = \|A\| \).

**Theorem 5.** Suppose \( A : D(A) \subseteq X \to Y \) is a densely defined operator. Then \( A^* \) is a closed operator.

**Proof.** Let \( (f_n) \) in \( D(A^*) \) such that \( f_n \to f \in Y^* \) and \( A^* f_n \to g \in X^* \). Since
\[
\langle A^* f_n, x \rangle = \langle f_n, Ax \rangle \quad \forall \ x \in D(A), \ n \in \mathbb{N},
\]
we have
\[
\langle g, x \rangle = \langle f, Ax \rangle \quad \forall \ x \in D(A).
\]
Hence, \( g \in D(A^*) \) and \( A^* f = g \). \( \square \)

The following example shows that the domain of the adjoint need not be dense even if \( A \) is a closed operator.

**Example 6.** Consider the Banach space \( \ell^1 \) and
\[
D := \{(\alpha_n) \in \ell^1 : (n\alpha_n) \in \ell^1 \}.
\]

Define
\[
A(\alpha_n) = (n\alpha_n), \quad (\alpha_n) \in D.
\]

Then \( A \) is a closed densely define operator: Since \( c_0 \subseteq D \), it follows that \( D \) is dense in \( \ell^1 \). To see that \( A \) is a closed operator, note first that \( A \) is surjective and bounded below:
\[
(\beta_n) \in \ell^1 \implies (\alpha_n) = (\beta_n/n) \in \ell^1 \in D, \ A(\alpha_n) = (\beta_n).,
\]
\[\|A(\alpha_n)\|_1 = \sum_n n|\alpha_n| \geq \sum_n |\alpha_n| \geq \|(\alpha_n)\|_1.\]

Hence, \(A^{-1}\) is continuous so that it is closed, and hence its inverse, which is \(A\), is also a closed operator.

But, the domain of \(A^*\) is not dense: For \((\beta_n) \in \ell^\infty\),
\[(\beta_n) \in D(A^*) \iff \exists (\gamma_n) \in \ell^1 \text{ such that } \langle (\gamma_n), (\alpha_n) \rangle = \langle (\beta_n), A(\alpha_n) \rangle \ \forall \ (\alpha_n) \in D.\]

Note that
\[\langle (\beta_n), A(\alpha_n) \rangle = \langle (\beta_n), (n\alpha_n) \rangle = \sum_n n\alpha_n\beta_n.\]

Thus,
\[\langle (\gamma_n), (\alpha_n) \rangle = \langle (\beta_n), A(\alpha_n) \rangle \iff \sum_n \alpha_n\gamma_n = \sum_n n\alpha_n\beta_n.\]

Hence, taking \((\alpha_n) = e_j\),
\[(\beta_n) \in D(A^*) \implies \gamma_j = j\beta_j \ \forall \ j \in \mathbb{N}.\]

Thus,
\[D(A^*) \subseteq \{(\beta_n) \in \ell^\infty : (n\beta_n) \in \ell^1 \} \subseteq c_0.\]

Hence, \(D(A^*)\) is not dense.

The following is a modified form of a theorem in Kato ([2], Theorem 5.29).

**Theorem 7.** Suppose \(A : D(A) \subseteq X \to Y\) is a closed densely defined operator and \(Y\) is a reflexive space. Then \(A^* : D(A^*) \subseteq Y^* \to X^*\) is a closed densely defined operator.

**Proof.** By Theorem 1, \(A^*\) is a closed operator. Hence it remains to proof that \(D(A^*)\) is dense. Suppose \(D(A^*)\) is not dense in \(Y^*\). Then there exists \(\varphi \in Y^{**}\) such that
\[\|\varphi\| = 1, \quad \varphi(f) = 0 \ \forall \ f \in D(A^*).\]

Since \(Y\) is reflexive, there exists \(y_0 \in Y\) such that
\[\|y_0\| = \|\varphi\|, \quad \varphi(f) = f(y_0) \ \forall \ f \in Y^*.\]

In particular,
\[\|y_0\| = 1, \quad f(y_0) = 0 \ \forall \ f \in D(A^*).\]

Now, \((0, y_0) \not\in G(A)\). Since \(G(A)\) is a closed subspace of \(X \times Y\), \(\exists F \in (X \times Y)^*\) such that \(F(0, y_0) \neq 0\) and \(F(x, Ax) = 0\) for all \(x \in D(A)\). Let \(f(y) = F(0, y)\). Then \(f \in Y^*\) and for \(x \in D(A)\),
\[f(Ax) = f(Ax) = F(0, Ax) = F(x, Ax) - F(x, 0) = -F(x, 0) = \langle g, x \rangle,\]
where \(g\) defined by \(g(x) = -F(x, 0)\) belongs to \(X^*\). Hence, \(f \in D(A^*)\). But, then by (i), \(f(y_0) = 0\). This is a contradiction since \(f(y_0) = F(0, y_0) \neq 0\). \(\square\)
It can be easily proved (see, eg. Nair [1]) that

- If $A : D(A) \subseteq X \to Y$ is a closed operator which is also continuous, then $D(A)$ is closed in $X$.

Hence, together with closed graph theorem, we obtain

**Theorem 8.** Suppose $A : D(A) \subseteq X \to Y$ is a closed densely defined operator. Then $A$ is continuous if and only if $D(A) = X$, and in that case $D(A^*) = Y^*$ and $A^* : Y^* \to X^*$ is continuous.

**Remark 9.** (i) Analogous definitions and results hold if we take dual spaces in place of adjoint spaces.

(ii) Suppose $X$ is a Hilbert space. For $f \in X^*$, let $\tilde{f}(x) = \overline{f(x)}$. Then we see that $\tilde{f} \in X'$, and hence by Riesz representation theorem, there exists a unique $u \in X$ such that

$$\tilde{f}(x) = \langle x, u \rangle_X \quad \forall x \in X.$$ 

Thus,

$$\langle f, x \rangle = f(x) = \langle u, x \rangle_X \quad \forall x \in X.$$ 

Hence, for every $f \in X^*$, there exists a unique $z_f \in X$ such that $\langle f, x \rangle = \langle z_f, x \rangle_X$ for all $x \in X$, and the map $f \mapsto z_f$ is a surjective linear isometry.

Suppose $A : D(A) \subseteq X \to Y$ is a densely defined operator between Hilbert spaces $X$ and $Y$. Then, in view of Remark 9 (ii),

$$\langle z_{A^* f}, x \rangle_X = \langle A^* f, x \rangle = \langle f, A x \rangle = \langle z_f, A x \rangle_Y \quad \forall x \in D(A), \, f \in D(A^*).$$

For $u \in Y$, let $f_u \in Y^*$ be defined by $f_u(y) = \langle u, y \rangle$, $y \in Y$. Then $z_{f_u} = u$. Thus, we obtain

$$\langle u, A x \rangle_Y = \langle z_{f_u}, A x \rangle_Y = \langle f_u, A x \rangle = \langle A^* f_u, x \rangle = \langle z_{A^* f_u}, x \rangle_X$$

for all $u \in Y$ such that $f_u \in D(A^*)$ and $x \in D(A)$. Let us define a linear operator $B : D(B) \subseteq Y \to X$ such that

$$D(B) = \{ u \in Y : f_u \in D(A^*) \},$$

and

$$Bu = z_{A^* f_u}.$$ 

Note that

$$\{ u \in Y : f_u \in D(A^*) \} = \{ u \in Y : x \mapsto \langle f_u, A x \rangle \text{ continuous} \} = \{ u \in Y : x \mapsto \langle u, A x \rangle_Y \text{ continuous} \}$$
Thus, 
\[ D(B) = \{ u \in Y : x \mapsto \langle u, Ax \rangle_Y \text{ continuous} \} \]
and 
\[ \langle u, Ax \rangle_Y = \langle Bu, x \rangle_X \quad \forall x \in D(A), u \in D(B). \]

**Definition 10.** If \( X \) and \( Y \) are Hilbert spaces, then the operator \( B : D(B) \subseteq Y \to X \) with 
\[ D(B) = \{ u \in Y : x \mapsto \langle u, Ax \rangle_Y \text{ continuous} \} \]
and 
\[ \langle u, Ax \rangle_Y = \langle Bu, x \rangle_X \quad \forall x \in D(A), u \in D(B) \]
is called the **adjoint** of \( A \).

If we denote by \( J_X : X^* \to X \) the map which takes \( f \in X^* \) to its Riesz representer \( z_f \), then we have 
\[ Bu = z_{A^*f_u} = J_X(A^*f_u) = J_XA^*J_Y^{-1}u \quad \forall u \in D(B). \]
Thus 
\[ B = J_XA^*J_Y^{-1} \quad \text{on} \quad D(B), \]
\[ A^* = J_X^{-1}BJ_Y \quad \text{on} \quad D(A^*). \]

In view of the above observations, (abusing the notation) we use the notation \( A^* \) for \( B \) also.

**References**


DEPARTMENT OF MATHEMATICS, I.I.T. MADRAS, CHENNAI-600 036, INDIA

_E-mail address: mtnair@iitm.ac.in_