Existence of Eigenvalues for Self-Adjoint Operators: 
A Simple Proof

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1 Introduction

Let $K$ denote either $\mathbb{R}$ or $\mathbb{C}$ depending on the context in which the discussion takes place, and $X$ be a finite vector space over $K$. Let $A : X \rightarrow X$ be a linear operator. Recall that $\lambda \in K$ is an eigenvalue of $A$ if there exists a non zero $x \in X$ such that

$$Ax = \lambda x,$$

and in that case $x$ is called an eigenvector of $A$ corresponding to the eigenvalue $\lambda$.

It is well known that if $K = \mathbb{C}$, then every $A : X \rightarrow X$ has at least one eigenvalue. This fact is normally proved in matrix theory and linear algebra using the concept of determinants. A determinant-free proof of this result is given in [1] (See also, [2]. However, if $K = \mathbb{R}$, then $A$ need not have any eigenvalue. In case $X$ is an inner product space and $A$ is a self-adjoint operator, i.e.,

$$\langle Ax, y \rangle = \langle x, Ay \rangle \quad \forall x, y \in X,$$

then it does have an eigenvalue even for the case of $K = \mathbb{R}$. The author has given a determinant-free proof for this fact as well (cf. [3]). In this note we give another proof for the same using a different line of arguments.

First we consider a few elementary facts: Let $X$ be an $n$- dimensional linear space over $K$ and $\mathcal{U} = \{u_1, \ldots, u_n\}$ be a basis for $X$. Let $A : X \rightarrow X$ is a linear operator
and let \([A]\) be the matrix representation of \(A\) w.r.t. \(U\), that is, \([A] = (a_{ij})\) is a matrix such that
\[
Au_j = \sum_{i=1}^{n} a_{ij} u_i.
\]
Note that if \(x = \sum_{j=1}^{n} \alpha_j u_j\), then
\[
Ax = \sum_{i=1}^{n} \left( \sum_{j=1}^{n} a_{ij} \alpha_j \right) u_i.
\]
Hence, for \(x = \sum_{j=1}^{n} \alpha_j u_j\) and \(\tilde{\alpha} := [\alpha_1, \ldots, \alpha_n]^T\),
\[
Ax = \lambda x \iff \sum_{j=1}^{n} a_{ij} \alpha_j = \lambda \alpha_i, \quad i = 1, \ldots, n \iff [A] \tilde{\alpha} = \lambda \tilde{\alpha}.
\]

## 2 Existence of Eigenvalues

Suppose \(X\) is an \(n\)-dimensional inner product space over \(\mathbb{K}\) and \(A : X \to X\) is a self adjoint operator. We already know (cf. [1], [2]) that if \(\mathbb{K} = \mathbb{C}\) then \(A\) has an eigenvalue, and since \(A\) is self-adjoint, every eigenvalue of \(A\) has to be real. So it is enough to consider the case of \(\mathbb{K} = \mathbb{R}\).

**THEOREM 2.1** Let \(X\) be a finite dimensional inner product space over \(\mathbb{R}\) and \(A : X \to X\) be a self adjoint operator. Then \(A\) has an eigenvalue.

We shall derive the proof of the above theorem from the following lemma.

**LEMMA 2.2** Suppose \(M\) is a real \(n \times n\) matrix. If \(M\), as a linear operator on \(\mathbb{C}^n\), has a real eigenvalue \(\lambda\), then there exists a nonzero \(\tilde{\alpha} \in \mathbb{R}^n\) such that \(M\tilde{\alpha} = \lambda \tilde{\alpha}\).

**Proof.** Consider \(M\) as a linear operator on \(\mathbb{C}^n\). Suppose \(M\) has an eigenvalue say \(\lambda \in \mathbb{C}\). Let \(\tilde{x} \in \mathbb{C}^n\) be a corresponding eigenvector. We may write \(\tilde{x} = \tilde{u} + i\tilde{v}\) with \(\tilde{u}, \tilde{v} \in \mathbb{R}^n\). Now,
\[
M\tilde{x} = M(\tilde{u} + i\tilde{v}) = M\tilde{u} + i[A]\tilde{v},
\]
\[
\lambda \tilde{x} = \lambda(\tilde{u} + i\tilde{v}) = \lambda \tilde{u} + i\lambda \tilde{v},
\]
so that using the fact that $M$ is a matrix with real entries, $\lambda \in \mathbb{R}$, and the relation $M\tilde{x} = \lambda \tilde{x}$, we have

$$M\tilde{u} = \lambda \tilde{u}, \quad M\tilde{v} = \lambda \tilde{v}.$$ 

Since $\tilde{x}$ is a non-zero vector in $\mathbb{C}^n$ at least one of $\tilde{u}$ and $\tilde{v}$ must be a non-zero. Thus we have a non-zero vector $\tilde{\alpha} \in \{\tilde{u}, \tilde{v}\}$ in $\mathbb{R}^n$ such that $M\tilde{\alpha} = \lambda \tilde{\alpha}$. □

**Proof of Theorem.** Suppose dim($X$) = $n$ and $\mathcal{U} = \{u_1, \ldots, u_n\}$ is a basis for $X$. Let $[A]$ be the matrix representation of $A$ w.r.t. $\mathcal{U}$. Now, $[A]$ can be considered as linear operator on $\mathbb{C}^n$. Hence $[A]$ has an eigenvalue say $\lambda \in \mathbb{C}$. Since $[A]$ is a self adjoint operator on $\mathbb{C}^n$, $\lambda$ must be real. Therefore, by the above lemma, there exists a nonzero $\tilde{\alpha} = [\alpha_1, \ldots, \alpha_n]^T \in \mathbb{R}^n$ such that

$$[A]\tilde{\alpha} = \lambda \tilde{\alpha}.$$ 

Consequently,

$$Ax = \lambda x$$

where $x := \sum_{j=1}^n \alpha_j u_j$. Thus, $A$ has an eigenvalue. □

**References**

