1 Introduction

Many of the inverse problems that occur in science and engineering are ill-posed, in the sense that, a unique solution that depends continuously on the data does not exist. A typical example of an ill-posed equation that often occurs in practical problems such as in geological prospecting, computer tomography, steel industry etc., is the Fredholm integral equation of the first kind (cf. [6], [2], [8]). Many such problems can be put in the form of an operator equation

\[ Ax = y, \quad (1.1) \]

where \( A : X \to Y \) is a bounded linear operator between Hilbert spaces \( X \) and \( Y \) with its range \( R(A) \) not closed in \( Y \).

Regularization methods are to be employed for obtaining a stable approximate solution for an ill-posed problem. Tikhonov regularization is a simple and widely used procedure to obtain stable approximate solutions to an ill-posed operator equation (1.1). In order to
improve the error estimates available in Tikhonov regularization, Natterer [17] carried out error analysis in the frame work of Hilbert scales. Subsequently, many authors extended, modified and generalized Natterer’s work to obtain error bounds under various contexts (cf. Neubauer [18], Hegland [7], Schröter and Tautenhahn [20], Mair [12], Nair, Hegland and Anderssen [16], Nair [13] and [14]). Finite dimensional realizations of the Hilbert scales approach has been considered by Engl and Neubauer [3].

If \( Y = X \) and \( A \) itself is a positive self–adjoint operator, then the simplified regularization introduced by Lavrentiev is better suited than Tikhonov regularization in terms of speed of convergence and condition numbers of the resulting equations in the case of finite dimensional approximations (cf. Schock [19]).

In [4], the authors introduced the Hilbert scales variant of the simplified regularization and obtained error estimates under a priori and a posteriori parameter choice strategies which are optimal in the sense of “best possible worst error” with respect to certain source set. Recently (cf. [5]), the authors considered a new discrepancy principle yielding optimal rates which does not involve certain restrictive assumptions as in [4]. The purpose of this paper is to obtain a finite dimensional realization of the results in [5].

2 Preliminaries

Let \( H \) be a Hilbert space and \( A : H \to H \) be a positive, bounded self-adjoint operator on \( H \). The inner product and the corresponding norm are denoted by \( \langle \cdot, \cdot \rangle \) and \( \| \cdot \| \) respectively. Recall that \( A \) is said to be a positive operator if \( \langle Ax, x \rangle \geq 0 \) for every \( x \in H \). For \( y \in R(A) \), the range of \( A \), consider the operator equation

\[
Ax = y. \tag{2.1}
\]

Let \( \hat{x} \) be the minimal norm solution of (2.1). It is well known that if \( R(A) \) is not closed in \( H \), then the problem of solving (2.1) for \( \hat{x} \) is ill–posed, in the sense that small perturbations in the data \( y \) can cause large deviations in the solution. A prototype of an ill-posed equation (2.1) is an integral equation of the first kind,

\[
\int_0^1 k(\xi, t)x(t) \, dt = y(\xi), \quad 0 \leq \xi \leq 1,
\]

where \( k(\cdot, \cdot) \) is a non-degenerate kernel which is square integrable, i.e.,

\[
\int_0^1 \int_0^1 |k(\xi, t)|^2 \, dt \, d\xi < \infty,
\]
satisfying $k(\xi, t) = k(t, \xi)$ for all $\xi, t$ in $[0, 1]$, and such that the eigenvalues of the corresponding integral operator $A : L^2[0, 1] \to L^2[0, 1]$,

$$(Ax)(\xi) = \int_0^1 k(\xi, t)x(t) \, dt, \quad 0 \leq \xi \leq 1, \quad (2.2)$$

are all non-negative (cf. [15]). For example, one of the important ill-posed problems which arise in applications is the \textit{backward heat equation} problem: The problem is to determine the initial temperature $\varphi_0 := u(\cdot, 0)$ from the measurements of the final temperature $\varphi_T := u(\cdot, T)$, where $u(\xi, t)$ satisfies

$$u_t - u_{\xi\xi} = 0, \quad (\xi, t) \in (0, 1) \times (0, T)$$

$$u(0, t) = u(1, t) = 0 \quad t \in [0, T].$$

We recall from elementary theory of partial differential equations that the solution $u(\xi, t)$ of the above heat equation is given by (cf. Weinberger [23])

$$u(\xi, t) = \sum_{n=1}^{\infty} \hat{\varphi}_0(n)e^{-n^2\pi^2 t}\sin(n\pi \xi),$$

where $\hat{\varphi}_0(n)$ for $n \in \mathbb{N}$ are the Fourier coefficients of the initial temperature $\varphi_0(\xi) := u(\xi, 0)$. Hence,

$$u(\xi, T) = \sum_{n=1}^{\infty} \hat{\varphi}_0(n)e^{-n^2\pi^2 T}\sin(n\pi \xi).$$

The above equation can be written as

$$\varphi_T(s) = \sum_{n=1}^{\infty} e^{-n^2\pi^2 T}\langle \varphi_0, u_n \rangle u_n(\xi) \quad \text{with} \quad u_n(\xi) = \sqrt{2}\sin(n\pi \xi).$$

Thus the problem is to solve the operator equation

$$A\varphi_0 = \varphi_T,$$

where $A : L^2[0, 1] \to L^2[0, 1]$ is the operator defined by

$$(A\varphi)(\xi) = \sum_{n=1}^{\infty} e^{-n^2\pi^2 T}\langle \varphi, u_n \rangle u_n(\xi) = \int_0^1 k(\xi, t)\varphi(t) \, dt, \quad 0 \leq \xi \leq 1,$$
where
\[ k(\xi, t) := \sum_{n=1}^{\infty} e^{-n^2 \pi^2 T} u_n(\xi) u_n(t). \]

Note that the above integral operator is compact, positive and self-adjoint with positive eigenvalues \( e^{-n^2 \pi^2 T} \) and corresponding eigenvectors \( u_n(\cdot) \) for \( n \in \mathbb{N} \).

For considering the regularization of the equation (2.1) in the setting of Hilbert scales, we consider a Hilbert scale \( \{H_t\}_{t \in \mathbb{R}} \) generated by a strictly positive operator \( L : D(L) \to H \) with its domain \( D(L) \) dense in \( H \) satisfying
\[ \|Lx\| \geq \|x\|, \quad x \in D(L). \]

By the operator \( L \) being strictly positive we mean that \( \langle Lx, x \rangle > 0 \) for all nonzero \( x \in H \).

Recall (cf. [9]) that the space \( H_t \) is the completion of \( D := \bigcap_{k=0}^{\infty} D(L^k) \) with respect to the norm \( \|x\|_t \), induced by the inner product
\[ \langle u, v \rangle_t = \langle L^t u, L^t v \rangle, \quad u, v \in D. \]

Moreover, if \( \beta \leq \gamma \), then the embedding \( H_\gamma \hookrightarrow H_\beta \) is continuous, and therefore the norm \( \|\cdot\|_\beta \) is also defined in \( H_\gamma \) and there is a constant \( c_{\beta, \gamma} \) such that
\[ \|x\|_{\beta} \leq c_{\beta, \gamma} \|x\|_\gamma \quad \forall x \in H_\beta. \] (2.3)

An important inequality that we require in the analysis is the interpolation inequality,
\[ \|x\|_\lambda \leq \|x\|_r^{\theta} \|x\|_t^{1-\theta}, \quad x \in H_t, \] (2.4)
where
\[ r \leq \lambda \leq t \quad \text{and} \quad \theta = \frac{t - \lambda}{t - r}, \]
and the moment inequality
\[ \|B^u x\| \leq \|B^v x\|^{\frac{u}{v}} \|x\|^{1-\frac{u}{v}}, \quad 0 \leq u \leq v, \] (2.5)
where \( B \) is a positive selfadjoint operator (cf. [2]).

We assume that the ill-posed nature of the operator \( A \) is related to the Hilbert scale \( \{H_t\}_{t \in \mathbb{R}} \) according to the relation
\[ c_1 \|x\|_{-a} \leq \|Ax\| \leq c_2 \|x\|_{-a}, \quad x \in H, \] (2.6)
for some positive reals \( a, c_1 \) and \( c_2 \).
For the example of the integral operator considered in (2.2), one may take $L$ to be defined by

$$Lx := \sum_{j=1}^{\infty} j^2 \langle x, u_j \rangle u_j,$$  \hspace{1cm} (2.7)

where $u_j(t) := \sqrt{2} \sin(j\pi t)$, $j \in \mathbb{N}$ with domain of $L$ as

$$D(L) := \{ x \in L^2[0, 1] : \sum_{j=1}^{\infty} j^4 |\langle x, u_j \rangle|^2 < \infty \}.$$  \hspace{1cm} (2.8)

In this case, it can be seen that, $H_t = \{ x \in L^2[0, 1] : \sum_{j=1}^{\infty} j^{4t} |\langle x, u_j \rangle|^2 < \infty \}$ and the constants $a$, $c_1$ and $c_2$ in (2.6) are given by $a = 1$, $c_1 = c_2 = 1/\pi^2$ (see Schröter and Tautenhahn [20], Section 4).

The regularized approximation of $\hat{x}$, considered in [4] is the solution of the well–posed equation

$$(A + \alpha L^s)x_\alpha = y, \quad \alpha > 0,$$  \hspace{1cm} (2.9)

where $s$ is a fixed non–negative real number. Note that, if $D(L) = X$ and $L = I$, then the above procedure is the simplified or Lavrentiev regularization.

Suppose the data $y$ is known only approximately, say $\tilde{y}$ in place of $y$ with $\|y - \tilde{y}\| \leq \delta$ for a known error level $\delta > 0$. Then, in place of (2.9), we have

$$(A + \alpha L^s)\tilde{x}_\alpha = \tilde{y}.$$  \hspace{1cm} (2.10)

It can be seen that the solution $\tilde{x}_\alpha$ of the above equation is the unique minimizer of the function

$$x \mapsto \langle Ax, x \rangle - 2\langle \tilde{y}, x \rangle + \alpha \langle L^s x, x \rangle, \quad x \in D(L).$$  \hspace{1cm} (2.11)

One of the crucial results for proving the results in [4] and [5] as well as the results in this paper is the following proposition, where the functions $f$ and $g$ are defined by

$$f(t) = \min\{c_1^t, c_2^t\}, \quad g(t) = \max\{c_1^t, c_2^t\}, \quad t \in \mathbb{R}, \quad |t| \leq 1,$$  \hspace{1cm} (2.12)

respectively, with $c_1, c_2$ as in (2.6).

**PROPOSITION 2.1** (cf. [4], Proposition 3.1) For $s > 0$ and $|\nu| \leq 1$,

$$f(\nu/2)\|x\|_{-\nu(s+a)/2} \leq \|A_s^{\nu/2} x\| \leq g(\nu/2)\|x\|_{-\nu(s+a)/2}, \quad x \in H,$$

where $A_s = L^{-s/2}AL^{-s/2}$.  \hspace{1cm} (5)
Using the above proposition the following result has been proved in George and Nair [4].

**THEOREM 2.2** (cf. Theorem 3.2 [4]) Suppose \( \hat{x} \in H_t \), \( 0 < t \leq s + a \) and \( \alpha > 0 \), and \( \hat{x}_\alpha \) is as in (2.10). Then

\[
\|\hat{x} - \hat{x}_\alpha\| \leq \phi(s, t)\alpha^{t/(s+a)} \|\hat{x}\|_t + \psi(s)\alpha^{-a/(s+a)}\delta, 
\]

where

\[
\phi(s, t) = \frac{g \left( \frac{s - 2t}{2s + 2a} \right)}{f \left( \frac{s}{2s + 2a} \right)} \quad \text{and} \quad \psi(s) = \frac{g \left( \frac{s}{2s + 2a} \right)}{f \left( \frac{s}{2s + 2a} \right)}. 
\]

In particular, if \( \alpha = c_0 \delta^{(s+a)/(t+a)} \) for some constant \( c_0 > 0 \), then

\[
\|\hat{x} - \hat{x}_\alpha\| \leq \eta(s, t)\delta^{t/(t+a)},
\]

where

\[
\eta(s, t) = \max \{ \phi(s, t)\|x\|_t c_0^{t/(t+a)}, \psi(s) c_0^{-a/(s+a)} \}. 
\]

For proposing a finite dimensional realization, we consider a family \( \{ S_h : h > 0 \} \) of finite dimensional subspaces of \( H_k \) for some \( k \geq s \), and consider the minimizer \( \hat{x}_{\alpha,h} \) of the map defined in (2.11) when \( x \) varies over \( S_h \). Equivalently, \( \hat{x}_{\alpha,h} \) is the unique element in \( S_h \) satisfying the equation

\[
\langle (A + \alpha L^*)\hat{x}_{\alpha,h}, \varphi \rangle = \langle \hat{y}, \varphi \rangle \quad \forall \varphi \in S_h. 
\]

As in Engl and Neubauer [3], we assume the following approximation properties for \( S_h \).

There exists a constant \( \kappa > 0 \) such that for every \( u \in H_r \) with \( r > k \geq s \),

\[
\inf \{ \|u - \varphi\|_k : \varphi \in S_h \} \leq \kappa h^{r-k}\|u\|_r, \quad h > 0. \tag{2.15}
\]

As already exemplified in [3], the above assumption is general enough to include a wide variety of approximations spaces such as spline spaces and finite element spaces.

We shall also make use of the following result from Engl and Neubauer ([3], Lemma 2.2).

**LEMMA 2.3** Under the assumption (2.15), there exists a constant \( c > 0 \) such that for every \( u \in H_s \) and \( h > 0 \),

\[
\inf_{\varphi \in S_h} \{ h^{-a/2}\|u - \varphi\|_{-a/2} + h^{s/2}\|u - \varphi\|_{s/2} \} \leq ch^s\|u\|_s.
\]
3 General Error Estimates

For a fixed $s > 0$, let $\tilde{x}_x$ and $\tilde{x}_{x,h}$ be as in (2.10) and (2.14) respectively. We shall obtain estimate for $\|\tilde{x}_x - \tilde{x}_{x,h}\|$ so that we get an estimate for $\|\hat{x} - \tilde{x}_{x,h}\|$ using Theorem 2.2 and the relation

$$\|\hat{x} - \tilde{x}_{x,h}\| \leq \|\hat{x} - \tilde{x}_x\| + \|\tilde{x}_x - \tilde{x}_{x,h}\|.$$  

In view of the interpolation inequality (2.4), by taking $\rho = -a/2$, $\tau = s/2$ and $\lambda = 0$ in (2.4), we get

$$\|x\| \leq \|x\|_{s/2}^{s/(s + a)} \|x\|_{s/2}^{a/(s + a)}, \quad x \in H_{s/2}.$$  

(3.1)

Thus, we can deduce an estimate for $\|\tilde{x}_x - \tilde{x}_{x,h}\|$ once we have estimates for $\|\tilde{x}_x - \tilde{x}_{x,h}\|_{-a/2}$ and $\|\tilde{x}_x - \tilde{x}_{x,h}\|_{s/2}$. For this purpose, we first prove the following.

**Lemma 3.1** Let $\tilde{x}_x$ and $\tilde{x}_{x,h}$ are as in (2.10) and (2.14) respectively. Then

$$\|A^{1/2}(\tilde{x}_x - \tilde{x}_{x,h})\|^2 + \alpha \|\tilde{x}_x - \tilde{x}_{x,h}\|^2_{s/2} = \inf_{\varphi \in S_h} \left\{ \|A^{1/2}(\tilde{x}_x - \varphi)\|^2 + \alpha \|\tilde{x}_x - \varphi\|^2_{s/2} \right\}.$$  

**Proof.** It can be seen (cf. [16]) that

$$\langle u, v \rangle_* := \langle Au, v \rangle + \alpha \langle L^* u, v \rangle, \quad u, v \in D(L),$$

defines a complete inner product on $D(L)$. Let $\| \cdot \|_*$ be the norm induced by $\langle \cdot, \cdot \rangle_*$, that is,

$$\|u\|_* = (\langle Au, u \rangle + \alpha \langle L^* u, u \rangle)^{1/2} = (\|A^{1/2} u\|^2 + \alpha \|u\|^2_{s/2})^{1/2}.$$  

Let $X$ be the space $D(L)$ with the inner product $\langle \cdot, \cdot \rangle_*$, and $P_h$ be the orthogonal projection of $X$ onto the space $S_h$. Then from equations (2.10) and (2.14) we have

$$\langle (A + \alpha L^*)(\tilde{x}_x - \tilde{x}_{x,h}), \varphi \rangle = 0 \quad \forall \varphi \in S_h.$$  

That is

$$\langle \tilde{x}_x - \tilde{x}_{x,h}, \varphi \rangle_* = 0 \quad \forall \varphi \in S_h.$$  

Hence

$$P_h(\tilde{x}_x - \tilde{x}_{x,h}) = 0,$$

so that

$$\|\tilde{x}_x - \tilde{x}_{x,h}\|_* = \inf_{\varphi \in S_h} \|\tilde{x}_x - \tilde{x}_{x,h} - \varphi\|_* = \inf_{\varphi \in S_h} \|\tilde{x}_x - \varphi\|_*.$$  

Now the result follows using the definition of $\| \cdot \|_*$. \qed

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Next we obtain estimate for $\|\tilde{x}_\alpha - \tilde{x}_{\alpha,h}\|$ using the estimates for $\|\tilde{x}_\alpha - \tilde{x}_{\alpha,h}\|_{-a/2}$ and $\|\tilde{x}_\alpha - \tilde{x}_{\alpha,h}\|_{s/2}$. We shall use the notation

$$A_s := L^{-s/2}AL^{-s/2}$$

and observe that for $\alpha > 0$,

$$(A + \alpha L^s)x = L^{s/2}(A_s + \alpha I)L^{s/2}x \quad \forall x \in H_s. \tag{3.2}$$

**THEOREM 3.2** Suppose $\tilde{x} \in H_t$ and assumption (2.15) holds, and let $\tilde{x}_\alpha$ and $\tilde{x}_{\alpha,h}$ are as in (2.10) and (2.14) respectively. Then

$$\|\tilde{x}_\alpha - \tilde{x}_{\alpha,h}\| \leq f(1/2)^{-\frac{s}{4+2a}} \max\{\mathcal{F}(s, a), \mathcal{G}(s, t, a)\} \Phi(s, h, \alpha)\alpha^{-\frac{s}{4+2a}} \left(\frac{\delta}{\alpha} + \alpha^{-\frac{a}{4+2a}}\right) h^s,$$

where $f$ and $g$ are as in (2.12), and

$$\mathcal{F}(s, a) = \frac{g \left(\frac{-s}{2+2a}\right)}{f \left(\frac{-s}{2+2a}\right)}, \quad \mathcal{G}(s, t, a) = \frac{g \left(\frac{s-2t}{2+2a}\right)}{f \left(\frac{s-2t}{2+2a}\right)} \|\tilde{x}\|_t,$$

$$\Phi(s, h, \alpha) = c \max\{g(1/2)h^{a/2}, \alpha^{1/2}h^{-s/2}\}.$$

**Proof.** First we prove

$$\|\tilde{x}_\alpha - \tilde{x}_{\alpha,h}\|_{-a/2} \leq \frac{1}{f(1/2)} \Phi(s, h, \alpha) h^s \|\tilde{x}_\alpha\|_s, \tag{3.3}$$

$$\|\tilde{x}_\alpha - \tilde{x}_{\alpha,h}\|_{s/2} \leq \Phi(s, h, \alpha)\alpha^{-1/2} h^s \|\tilde{x}_\alpha\|_s, \tag{3.4}$$

and

$$\|\tilde{x}_\alpha\|_s \leq \mathcal{F}(s, a)\alpha^{-1} \delta + \mathcal{G}(s, t, a)\alpha^{\frac{a}{4+2a}} \tag{3.5}$$

with $\mathcal{F}(s, a), \mathcal{G}(s, t, a)$ and $\Phi(s, h, \alpha)$ are as in the statement of the theorem.

By Lemma 3.1 and Proposition 2.1, it follows that

$$f(1/2)^2\|\tilde{x}_\alpha - \tilde{x}_{\alpha,h}\|_{-a/2}^2 + \alpha\|\tilde{x}_\alpha - \tilde{x}_{\alpha,h}\|_{s/2}^2 \leq \inf_{\varphi \in \mathcal{S}_h} \left\{g(1/2)^2\|\tilde{x}_\alpha - \varphi\|_{-a/2}^2 + \alpha\|\tilde{x}_\alpha - \varphi\|_{s/2}^2\right\}.$$

Note that

$$g(1/2)^2\|\tilde{x}_\alpha - \varphi\|_{-a/2}^2 + \alpha\|\tilde{x}_\alpha - \varphi\|_{s/2}^2 \leq \left[g(1/2)^2\|\tilde{x}_\alpha - \varphi\|_{-a/2} + \alpha^{1/2}\|\tilde{x}_\alpha - \varphi\|_{s/2}\right]^2.$$

But

$$g(1/2)^2\|\tilde{x}_\alpha - \varphi\|_{-a/2} + \alpha^{1/2}\|\tilde{x}_\alpha - \varphi\|_{s/2} \leq \omega_{h,a,s} \left[h^{-a/2}\|\tilde{x}_\alpha - \varphi\|_{-a/2} + h^{s/2}\|\tilde{x}_\alpha - \varphi\|_{s/2}\right],$$
where

\[ \omega(h, \alpha, s) := \max\{g(1/2)h^{a/2}, \alpha^{1/2}h^{-s/2}\}. \]

Hence, by Lemma 2.3, we have

\[ f(1/2)^2 \|\tilde{x}_\alpha - \tilde{x}_{\alpha,h}\|^2_{1/2} + \alpha \|\tilde{x}_\alpha - \tilde{x}_{\alpha,h}\|^2_{s/2} \leq (\omega(h, \alpha, s)c h^s \|\tilde{x}_\alpha\|)^2. \]

In particular,

\[ f(1/2)\|\tilde{x}_\alpha - \tilde{x}_{\alpha,h}\|_{-a/2} \leq \omega(h, \alpha, s)c h^s \|\tilde{x}_\alpha\|, \]

\[ \alpha^{1/2}\|\tilde{x}_\alpha - \tilde{x}_{\alpha,h}\|_{s/2} \leq \omega(h, \alpha, s)c h^s \|\tilde{x}_\alpha\|. \]

From these, we obtain (3.3) and (3.4).

Now, to prove (3.5), observe from (2.10) and (3.2) that

\[ \tilde{x}_\alpha = L^{-s/2}(A_s + \alpha I)^{-1}L^{-s/2}\tilde{y}. \]

By Proposition 2.1, taking \( \nu = -s/(s + a) \), we have

\[ \|L^{s/2}(A_s + \alpha I)^{-1}L^{-s/2}(\tilde{y} - y)\| \leq \frac{1}{f\left(-\frac{s}{2s+2a}\right)}\|A_s^{\frac{-s}{2s+2a}}(A_s + \alpha I)^{-1}L^{-s/2}(\tilde{y} - y)\| \]

\[ \leq \frac{\|A_s + \alpha I\|^{-1}}{f\left(-\frac{a}{2s+2a}\right)}\|A_s^{\frac{-s}{2s+2a}}L^{-s/2}(\tilde{y} - y)\| \]

\[ \leq \frac{\alpha^{-1}g\left(-\frac{a}{2s+2a}\right)}{f\left(-\frac{s}{2s+2a}\right)}\|L^{-s/2}(\tilde{y} - y)\|_{s/2} \]

so that

\[ \|L^{s/2}(A_s + \alpha I)^{-1}L^{-s/2}(\tilde{y} - y)\| \leq \mathcal{F}(s, a)\alpha^{-1}\delta. \]  \( \quad \text{(3.6)} \)

Since \( L^{-s/2}y = A_sL^{s/2}\tilde{x} \), we have

\[ \|L^{s/2}(A_s + \alpha I)^{-1}L^{-s/2}y\| \leq \frac{1}{f\left(-\frac{s}{2s+2a}\right)}\|A_s^{\frac{s}{2s+2a}}(A_s + \alpha I)^{-1}A_sL^{s/2}\tilde{x}\| \]

\[ \leq \frac{1}{f\left(-\frac{s}{2s+2a}\right)}\|(A_s + \alpha I)^{-1}A_s^{\frac{s}{2s+2a}}\|\|A_s^{\frac{s}{2s+2a}}L^{s/2}\tilde{x}\|, \]

where

\[ \|A_s^{\frac{s}{2s+2a}}L^{s/2}\tilde{x}\| \leq g\left(\frac{s - 2t}{2s + 2a}\right)\|\tilde{x}\|_t. \]

Since

\[ \|(A_s + \alpha I)^{-1}A_s^\tau\| \leq \alpha^{\tau-1}, \quad 0 < \tau \leq 1, \]

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it follows from the above relations that
\[
\|L_s^{s/2}(A_s + \alpha I)^{-1}L_s^{-s/2}y\| \leq \frac{g\left(\frac{s-2t}{2s+2a}\right)}{f\left(\frac{s-2t}{2s+2a}\right)} \|\hat{x}\|_t \alpha^{\frac{s}{s+a}} = \mathcal{G}(s, t, a)\alpha^{\frac{s}{s+a}}.
\] (3.7)

Thus, equations (3.6) and (3.7) give
\[
\|\tilde{x}_\alpha\|_s \leq \|L_s^{s/2}(A_s + \alpha I)^{-1}L_s^{-s/2}y\|
\]
(3.8)
\[
\leq \|L_s^{s/2}(A_s + \alpha I)^{-1}L_s^{-s/2}(\tilde{y} - y)\| + \|L_s^{s/2}(A_s + \alpha I)^{-1}L_s^{-s/2}y\|
\]
\[
\leq \mathcal{F}(s, a)\alpha^{-1}\delta + \mathcal{G}(s, t, a)\alpha^{\frac{s}{s+a}}.
\]

Now, the estimates (3.3) and (3.4) together with the interpolation inequality (3.1) give
\[
\|\tilde{x}_\alpha - \tilde{x}_{\alpha, h}\| \leq \|\tilde{x}_\alpha - \tilde{x}_{\alpha, h}\|_{\frac{s}{s+2}}\|\tilde{x}_\alpha - \tilde{x}_{\alpha, h}\|_{\frac{s}{s+2}}
\]
(3.9)
\[
\leq f(1/2)\alpha^{-\frac{s}{s+a+2}}\max\{\mathcal{F}(s, a), \mathcal{G}(s, t, a)\} \|	ilde{x}_\alpha\|_s.
\]

From this the result follows by making use of the estimate (3.5) for \(\tilde{x}_\alpha\).

\[\Box\]

4 A Priori Error Estimates

Now we choose the regularization parameter \(\alpha\) and discretization parameter \(h\) a priori depending on the noise level \(\delta\) such that optimal order \(O(\delta^{1/(t+a)})\) is yielded whenever \(\hat{x} \in H_t\).

**THEOREM 4.1** Suppose \(\hat{x} \in H_t\) with \(0 < t \leq s + a\) and assumption (2.15) holds. Suppose, in addition, that
\[
\alpha = c_0\delta^{\frac{s}{s+a}} \quad \text{and} \quad h = d_0\delta^{\frac{1}{t+a}}
\]
for some constants \(c_0, d_0 > 0\). Then, using the notations in Theorem 2.2 and Theorem 3.2,
\[
\|\hat{x} - \tilde{x}_{\alpha, h}\| \leq [\eta(s, t) + \xi(s, t)]\delta^{\frac{1}{t+a}},
\]
where
\[
\eta(s, t) = \max\left\{\phi(s, t)\|x\|_t c_0^{\frac{t}{s+a}}, \psi(s)\|x\|_t c_0^{\frac{t}{s+a}}\right\}
\]
and
\[
\xi(s, t) = c[f(1/2)]^{\frac{2s}{s+a}}d_0\left(c_0^{-1} + c_0^{\frac{t}{s+a}}\right)\max\{\mathcal{F}(s, a), \mathcal{G}(s, t, a)\} \max\{g(1/2)d_0^\frac{2}{t}, c_0^1d_0^\frac{2}{t}\}.
\]
Proof. Using the choice
\[ \alpha = c_0 \delta^{\frac{s}{t+s}} \quad \text{and} \quad h = d_0 \delta^{\frac{1}{t+s}}, \]
it is seen that
\[ \Phi(s, h, \alpha) = c c_0^{-a/2(s+a)} \max\{g(1/2)d_0^{a/2}, c_0^{1/2}d_0^{-s/2}\}, \]
and
\[ \delta \alpha^{-1} h^s = c_0^{-1} d_0^{s/2} \delta^{\frac{s}{t+s}}, \]
Therefore, by Theorem 3.2, we have
\[ \| \tilde{x}_\alpha - \tilde{x}_{\alpha, h} \| \leq \xi(s, t) \delta^{\frac{1}{t+s}}. \]
Also, from Theorem 2.2, we have
\[ \| \hat{x} - \tilde{x}_\alpha \| \leq \eta(s, t) \delta^{\frac{1}{t+s}}. \]
Thus the result follows from the inequality
\[ \| \hat{x} - \tilde{x}_\alpha \| \leq \| \hat{x} - \tilde{x}_\alpha \| + \| \tilde{x}_\alpha - \tilde{x}_{\alpha, h} \|. \]

Remark. We observe that the error bound obtained is of the same order as of Theorem 2.2, and this order is optimal with respect to the source set
\[ M_{\rho, t} = \{ x \in H_t : \| x \|_t \leq \rho \} \]
in the sense of best possible worst error (cf. [4]).

5 Discrepancy Principle

In this section we consider a discrepancy principle to choose the regularization parameter \( \alpha \) depending on the noise level \( \delta \) and the discretization parameter \( h \). This is a finite dimensional variant of the discrepancy principle considered in [5].

We assume throughout that \( y \neq 0 \). Suppose \( \tilde{y} \in H \) be such that
\[ \| y - \tilde{y} \| \leq \delta \quad (5.1) \]
for a known error level $\delta > 0$ and $P_h\tilde{y} \neq 0$ where $P_h$ is the orthogonal projection of $H$ onto $S_h$. We assume, throughout this section, that

$$\|A(P_h-I)\| \leq c_3h, \quad h > 0,$$

for some $c_3 > 0$, independent of $h$. Let

$$R_\alpha := (A_s + \alpha I)^{-1}.$$

We shall make use of the relation

$$\|R_\alpha A^s\| \leq \alpha \tau^{-1}, \quad 0 < \alpha < 1 \quad (5.3)$$

which follows from the spectral properties of the self adjoint operator $A_s, s > 0$.

Let $s, a$ be fixed positive real numbers. For $\alpha > 0$ and $x \in H$, consider the functions

$$F(\alpha, x) = \frac{\alpha \|R^3_\alpha A^{\frac{s}{s+2a}} L^\frac{2}{s} P_h x\|^2}{\|R^2_\alpha A^{\frac{s}{s+2a}} L^\frac{2}{s} P_h x\|^2}. \quad (5.4)$$

Note that, by assumption (2.6), $\|R^3_\alpha A^{\frac{s}{s+2a}} L^\frac{2}{s} P_h x\|$ is non zero for every $x \in H$ with $P_h x \neq 0$, so that the function $F(\alpha, x)$ is well defined for all such $x$. We observe that the assumption $P_h x \neq 0$ is satisfied for $x \neq 0$ and $h$ small enough, if $P_h x \to x$ as $h \to 0$ for every $x \in H$.

In the following we assume that $h$ is such that $P_h\tilde{y} \neq 0$.

In order to choose the regularization parameter $\alpha$, we consider the discrepancy principle

$$F(\alpha, \tilde{y}) = b\delta + dh, \quad (5.5)$$

for some $b, d > 0$. In the due course we shall make use of the relation

$$f \left( \frac{-s}{2s + 2a} \right) \|x\| \leq \|A_s^{\frac{s}{s+2a}} L^\frac{2}{s} x\| \leq g \left( \frac{-s}{2s + 2a} \right) \|x\| \quad (5.6)$$

which can be easily derived from Proposition 2.1.

First we prove the monotonicity of the function $F(\alpha, x)$ defined in (5.4).

**THEOREM 5.1** Let $x \in H$ be such that the function $\alpha \mapsto F(\alpha, x)$ for $\alpha > 0$ in (5.4) is well-defined. Then, $F(\cdot, x)$ is increasing and it is continuously differentiable with $F'(\alpha, x) \geq 0$ for all $\alpha > 0$. In addition

$$\lim_{\alpha \to 0} F(\alpha, x) = 0, \quad \lim_{\alpha \to \infty} F(\alpha, x) = \|A_s^{\frac{s}{s+2a}} L^\frac{2}{s} P_h x\|. \quad 12$$
\begin{proof}
Using the definition (5.4) of $F(\alpha, \cdot)$, we have

\[
\frac{\partial}{\partial \alpha} F(\alpha, x) = \frac{\partial}{\partial \alpha} \left( \frac{F^2(\alpha, x)}{2F(\alpha, x)} \right) = 2\alpha \left[ R_\alpha^2 A_s^{\frac{3}{2}+\frac{\alpha}{2}} L^{\frac{\alpha}{2}} P_h x \right]^2 \left[ R_\alpha^3 A_s^{\frac{3}{2}+\frac{\alpha}{2}} L^{\frac{\alpha}{2}} P_h x \right]^2 \frac{\partial}{\partial \alpha} \left[ \frac{\alpha}{2} \left( R_\alpha^3 A_s^{\frac{3}{2}+\frac{\alpha}{2}} L^{\frac{\alpha}{2}} P_h x \right)^2 \right] = 2\alpha \left\| R_\alpha^3 A_s^{\frac{3}{2}+\frac{\alpha}{2}} L^{\frac{\alpha}{2}} P_h x \right\|^3 \left[ \frac{\alpha}{2} \left( R_\alpha^3 A_s^{\frac{3}{2}+\frac{\alpha}{2}} L^{\frac{\alpha}{2}} P_h x \right)^2 \right] = \alpha^2 \left\| R_\alpha^3 A_s^{\frac{3}{2}+\frac{\alpha}{2}} L^{\frac{\alpha}{2}} P_h x \right\|^4 \frac{\partial}{\partial \alpha} \left[ \frac{\alpha}{2} \left( R_\alpha^3 A_s^{\frac{3}{2}+\frac{\alpha}{2}} L^{\frac{\alpha}{2}} P_h x \right)^2 \right] = \alpha \left\| R_\alpha^3 A_s^{\frac{3}{2}+\frac{\alpha}{2}} L^{\frac{\alpha}{2}} P_h x \right\|^3 \frac{\partial}{\partial \alpha} \left[ \frac{\alpha}{2} \left( R_\alpha^3 A_s^{\frac{3}{2}+\frac{\alpha}{2}} L^{\frac{\alpha}{2}} P_h x \right)^2 \right] = \frac{\alpha^2}{2} \left\| R_\alpha^3 A_s^{\frac{3}{2}+\frac{\alpha}{2}} L^{\frac{\alpha}{2}} P_h x \right\|^2 \left[ -3\alpha \left( R_\alpha^2 A_s^{\frac{3}{2}+\frac{\alpha}{2}} L^{\frac{\alpha}{2}} P_h x \right)^2 \right].\end{proof}

Let $\{E_\lambda : 0 \leq \lambda \leq a\}$ be the spectral family of $A_s$, where $a \geq \|A_s\|$. Then

\[
\frac{\partial}{\partial \alpha} \left( \alpha \left\| R_\alpha^3 A_s^{\frac{3}{2}+\frac{\alpha}{2}} L^{\frac{\alpha}{2}} P_h x \right\|^2 \right) = \frac{\partial}{\partial \alpha} \int_0^a \frac{\alpha}{\lambda^{\frac{\alpha}{2}}(\lambda + \alpha)^3} d\langle E_\lambda L^{\frac{\alpha}{2}} P_h x, L^{\frac{\alpha}{2}} P_h x \rangle = \int_0^a \left[ \frac{1}{\lambda^{\frac{\alpha}{2}}(\lambda + \alpha)^3} - \frac{3\alpha}{\lambda^{\frac{\alpha}{2}}(\lambda + \alpha)^4} \right] d\langle E_\lambda L^{\frac{\alpha}{2}} P_h x, L^{\frac{\alpha}{2}} P_h x \rangle = \left\| R_\alpha^3 A_s^{\frac{3}{2}+\frac{\alpha}{2}} L^{\frac{\alpha}{2}} P_h x \right\|^2 - 3\alpha \left\| R_\alpha^2 A_s^{\frac{3}{2}+\frac{\alpha}{2}} L^{\frac{\alpha}{2}} P_h x \right\|^2.\end{proof}

Similarly, we obtain

\[
\frac{\partial}{\partial \alpha} (\| R_\alpha^3 A_s^{\frac{3}{2}+\frac{\alpha}{2}} L^{\frac{\alpha}{2}} P_h x \|) = -4\| R_\alpha^2 A_s^{\frac{3}{2}+\frac{\alpha}{2}} L^{\frac{\alpha}{2}} P_h x \|^2.\end{proof}

Therefore from (5.7), by using (5.8) and (5.9), we get

\[
\frac{\partial}{\partial \alpha} F(\alpha, x) = \frac{\| R_\alpha^2 A_s^{\frac{3}{2}+\frac{\alpha}{2}} L^{\frac{\alpha}{2}} P_h x \|^2 \left\| R_\alpha^3 A_s^{\frac{3}{2}+\frac{\alpha}{2}} L^{\frac{\alpha}{2}} P_h x \right\|^2 \left[ -3\alpha \left( R_\alpha^2 A_s^{\frac{3}{2}+\frac{\alpha}{2}} L^{\frac{\alpha}{2}} P_h x \right)^2 \right] \| R_\alpha^2 A_s^{\frac{3}{2}+\frac{\alpha}{2}} L^{\frac{\alpha}{2}} P_h x \|^3}{\| R_\alpha^3 A_s^{\frac{3}{2}+\frac{\alpha}{2}} L^{\frac{\alpha}{2}} P_h x \|^3} + 2\alpha \left\| R_\alpha^3 A_s^{\frac{3}{2}+\frac{\alpha}{2}} L^{\frac{\alpha}{2}} P_h x \right\|^2 \left\| R_\alpha^2 A_s^{\frac{3}{2}+\frac{\alpha}{2}} L^{\frac{\alpha}{2}} P_h x \right\|^2 \| R_\alpha^2 A_s^{\frac{3}{2}+\frac{\alpha}{2}} L^{\frac{\alpha}{2}} P_h x \|^3}}{\| R_\alpha^2 A_s^{\frac{3}{2}+\frac{\alpha}{2}} L^{\frac{\alpha}{2}} P_h x \|^3} + 2\alpha \left\| R_\alpha^3 A_s^{\frac{3}{2}+\frac{\alpha}{2}} L^{\frac{\alpha}{2}} P_h x \right\|^2 \left\| R_\alpha^2 A_s^{\frac{3}{2}+\frac{\alpha}{2}} L^{\frac{\alpha}{2}} P_h x \right\|^2 \| R_\alpha^2 A_s^{\frac{3}{2}+\frac{\alpha}{2}} L^{\frac{\alpha}{2}} P_h x \|^3}}{\| R_\alpha^2 A_s^{\frac{3}{2}+\frac{\alpha}{2}} L^{\frac{\alpha}{2}} P_h x \|^3} + \alpha \left\| R_\alpha^2 A_s^{\frac{3}{2}+\frac{\alpha}{2}} L^{\frac{\alpha}{2}} P_h x \right\|^3 \frac{\partial}{\partial \alpha} \left[ \frac{\alpha}{2} \left( R_\alpha^3 A_s^{\frac{3}{2}+\frac{\alpha}{2}} L^{\frac{\alpha}{2}} P_h x \right)^2 \right] = \frac{\alpha^2}{2} \left\| R_\alpha^3 A_s^{\frac{3}{2}+\frac{\alpha}{2}} L^{\frac{\alpha}{2}} P_h x \right\|^2 \left[ -3\alpha \left( R_\alpha^2 A_s^{\frac{3}{2}+\frac{\alpha}{2}} L^{\frac{\alpha}{2}} P_h x \right)^2 \right].\end{proof}

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The above equation can be rewritten as

\[
\frac{\partial}{\partial \alpha} F(\alpha, x) = \frac{\|R_\alpha^3 A_s^{\frac{\alpha}{\alpha+2}} L^{\frac{\alpha}{\alpha+2}} P_h x\|^2}{\|R_\alpha^3 A_s^{\frac{\alpha}{\alpha+2}} L^{\frac{\alpha}{\alpha+2}} P_h x\|^3} - \frac{\alpha \|R_\alpha^3 A_s^{\frac{\alpha}{\alpha+2}} L^{\frac{\alpha}{\alpha+2}} P_h x\|^2}{\|R_\alpha^3 A_s^{\frac{\alpha}{\alpha+2}} L^{\frac{\alpha}{\alpha+2}} P_h x\|^3} \]

Since

\[
\|R_\alpha^3 A_s^{\frac{\alpha}{\alpha+2}} L^{\frac{\alpha}{\alpha+2}} P_h x\|^2 = \langle (A_s + \alpha I)^{-3} A_s^{\frac{\alpha}{\alpha+2}} L^{\frac{\alpha}{\alpha+2}} P_h x, A_s^{\frac{\alpha}{\alpha+2}} L^{\frac{\alpha}{\alpha+2}} P_h x \rangle,
\]

we see that

\[
\|R_\alpha^3 A_s^{\frac{\alpha}{\alpha+2}} L^{\frac{\alpha}{\alpha+2}} P_h x\|^2 = \langle (A_s + \alpha I)^{-3} A_s^{\frac{\alpha}{\alpha+2}} L^{\frac{\alpha}{\alpha+2}} P_h x, (A_s + \alpha I)^{-1} A_s^{\frac{\alpha}{\alpha+2}} L^{\frac{\alpha}{\alpha+2}} P_h x \rangle
\]

Also, we have

\[
\|R_\alpha^3 A_s^{\frac{\alpha}{\alpha+2}} L^{\frac{\alpha}{\alpha+2}} P_h x\|^4 = \left[ \langle R_\alpha^3 A_s^{\frac{\alpha}{\alpha+2}} L^{\frac{\alpha}{\alpha+2}} P_h x, R_\alpha^3 A_s^{\frac{\alpha}{\alpha+2}} L^{\frac{\alpha}{\alpha+2}} P_h x \rangle \right]^2
\]

\[
\leq \|R_\alpha^3 A_s^{\frac{\alpha}{\alpha+2}} L^{\frac{\alpha}{\alpha+2}} P_h x\|^2 \|R_\alpha^\frac{\alpha}{\alpha+2} A_s^{\frac{\alpha}{\alpha+2}} L^{\frac{\alpha}{\alpha+2}} P_h x\|^2.
\]

Hence,

\[
\frac{\partial}{\partial \alpha} (F(\alpha, x)) \geq 0.
\]

To prove the last part of the theorem we observe that,

\[
\alpha^2 \|R_\alpha^3 A_s^{\frac{\alpha}{\alpha+2}} L^{\frac{\alpha}{\alpha+2}} P_h x\| - F(\alpha, x) = \frac{\alpha^2 \|R_\alpha^3 A_s^{\frac{\alpha}{\alpha+2}} L^{\frac{\alpha}{\alpha+2}} P_h x\|^2 - \alpha \|R_\alpha^3 A_s^{\frac{\alpha}{\alpha+2}} L^{\frac{\alpha}{\alpha+2}} P_h x\|^2}{\|R_\alpha^3 A_s^{\frac{\alpha}{\alpha+2}} L^{\frac{\alpha}{\alpha+2}} P_h x\|^2}.
\]

We note that

\[
\alpha^2 \|R_\alpha^3 A_s^{\frac{\alpha}{\alpha+2}} L^{\frac{\alpha}{\alpha+2}} P_h x\|^2 = \alpha \langle R_\alpha^3 A_s^{\frac{\alpha}{\alpha+2}} L^{\frac{\alpha}{\alpha+2}} P_h x, \alpha R_\alpha A_s^{\frac{\alpha}{\alpha+2}} L^{\frac{\alpha}{\alpha+2}} P_h x \rangle
\]
and

$$\alpha \| R_\alpha^3 A_s^{\frac{r-4}{2s}} L^{\frac{r}{s}} P_h x \|^2 = \alpha \langle R_\alpha^3 A_s^{\frac{r-4}{2s}} L^{\frac{r}{s}} P_h x, A_s^{\frac{r-4}{2s}} L^{\frac{r}{s}} P_h x \rangle.$$  

Since

$$\alpha R_\alpha - I = A_s R_\alpha = R_\alpha A_s,$$

it follows that

$$\alpha^2 \| R_\alpha^2 A_s^{\frac{r-4}{2s}} L^{\frac{r}{s}} P_h x \| - F(\alpha, x) = \frac{-\alpha \langle R_\alpha^3 A_s^{\frac{r-4}{2s}} L^{\frac{r}{s}} P_h x, A_s R_\alpha A_s^{\frac{r-4}{2s}} L^{\frac{r}{s}} P_h x \rangle}{\| R_\alpha^2 A_s^{\frac{r-4}{2s}} L^{\frac{r}{s}} P_h x \|^2} \leq 0 \quad (5.10)$$

so that

$$F(\alpha, x) \geq \alpha^2 \| R_\alpha^2 A_s^{\frac{r-4}{2s}} L^{\frac{r}{s}} P_h x \| \geq \alpha^2 \frac{\| A_s^{\frac{r-4}{2s}} L^{\frac{r}{s}} P_h x \|}{\| A_s \| + \alpha}.$$  

Also, we have

$$F(\alpha, x) = \frac{\alpha \langle R_\alpha A_s^{\frac{r-4}{2s}} L^{\frac{r}{s}} P_h x, R_\alpha^2 A_s^{\frac{r-4}{2s}} L^{\frac{r}{s}} P_h x \rangle}{\| R_\alpha^2 A_s^{\frac{r-4}{2s}} L^{\frac{r}{s}} P_h x \|^2} \leq \alpha \| R_\alpha A_s^{\frac{r-4}{2s}} L^{\frac{r}{s}} P_h x \|. \quad (5.11)$$

Hence

$$\left( \frac{\alpha}{\| A_s \| + \alpha} \right)^2 \| A_s^{\frac{r-4}{2s}} L^{\frac{r}{s}} P_h x \| \leq F(\alpha, x) \leq \alpha \| R_\alpha A_s^{\frac{r-4}{2s}} L^{\frac{r}{s}} P_h x \|.$$  

From this we can conclude that

$$\lim_{\alpha \to 0} F(\alpha, x) = 0 \quad \text{and} \quad \lim_{\alpha \to \infty} F(\alpha, x) = \| A_s^{\frac{r-4}{2s}} L^{\frac{r}{s}} P_h x \|.$$  

This completes the proof. \(\square\)

For the next theorem, in addition to (5.1), we assume that the inexact data \(\tilde{y}\) satisfies the relation

$$\| A_s^{\frac{r-4}{2s}} L^{\frac{r}{s}} P_h \tilde{y} \| \geq b\delta + dh. \quad (5.12)$$

This assumptions is satisfied for small enough \(h\) and \(\delta\), if, for example,

$$(b + \tilde{f}(s))\delta + (d + c_3 \tilde{f}(s)\| \hat{x} \|) h \leq \tilde{f}(s)\| y \|,$$

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where \( \tilde{f}(s) = f(\frac{-s}{2s+2a}) \), because

\[
\|P_h \tilde{y}\| \geq \|y\| - \|(I-P_h)A\tilde{x}\| - \delta
\]

and by (5.6)

\[
\|A_{\frac{-s}{2s+2a}}^\tau L^{-\frac{s}{2s+2a}}P_h \tilde{y}\| \geq \tilde{f}(s)\|P_h \tilde{y}\|.
\]

Now the following theorem is a consequence of Theorem 5.1

**THEOREM 5.2** Assume that (5.1) and (5.12) are satisfied. Then there exist a unique \( \alpha := \alpha(\delta, h) \) satisfying

\[
F(\alpha, \tilde{y}) = b\delta + dh
\]  

(5.13)

In order to obtain an estimate for the error \( \|\tilde{x} - \tilde{x}_{\alpha,h}\| \) with the parameter choice strategy (5.13), we shall make use of (3.9). The next lemma gives an error estimate for \( \|\tilde{x}_\alpha\|_s \) in terms of \( \alpha = \alpha(\delta, h), \delta \) and \( h \).

**LEMMA 5.3** Let \( \alpha := \alpha(\delta, h) \) be the unique solution of (5.13). Then for any fixed \( \tau > 0 \),

\[
\|\tilde{x}_\alpha\|_s \leq c_4(\delta + h)^{\frac{s}{2s+2a}}\alpha^{-1},
\]

where

\[
c_4 \geq \max\{b + \tilde{g}(s), c + c_3\|\tilde{x}\|\tilde{g}(s)\|\tilde{y}\|
\]

with \( \tilde{g}(s) := g(\frac{s}{2s+2a}) \).

**Proof.** By (3.8) we have

\[
\|\tilde{x}_\alpha\|_s \leq \|L^\tau R_\alpha L^{-\frac{s}{2s+2a}}\tilde{y}\|
\]

\[
\leq \tilde{f}^{-1}(s)\alpha^{-1}\|\alpha R_\alpha A_{\frac{-s}{2s+2a}}^\tau L^\tau \tilde{y}\|.
\]

(5.14)

To obtain an estimate for \( \|\alpha R_\alpha A_{\frac{-s}{2s+2a}}^\tau L^\tau \tilde{y}\| \), we shall make use of the moment inequality (2.5). Precisely we use (2.5) with

\[
u = \tau, \quad v = 1 + \tau, \quad B = \alpha R_\alpha \quad x = \alpha^{-\tau} R_\alpha^{1-\tau} A_{\frac{-s}{2s+2a}}^\tau L^\tau \tilde{y}.
\]

Then, since

\[
\|x\| \leq \|\alpha A_{\frac{-s}{2s+2a}}^\tau L^\tau \tilde{y}\| \leq g\left(\frac{-s}{2s+2a}\right)\|\tilde{y}\|
\]

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we have
\[
\|\alpha R_\alpha A_s^{\frac{s}{2s+2a}} L^{-\frac{s}{2}} \hat{y}\| \leq \|B^T x\| \\
\leq \|B^{r+1} x\| \frac{\|x\|}{r+1} \\
= \|\alpha^2 R_\alpha^2 A_s^{\frac{s}{2s+2a}} L^{-\frac{s}{2}} \hat{y}\| \frac{\|g(\frac{-s}{2s+2a})\|\|\hat{y}\|}{r+1} \tag{5.15}
\]

Further by (5.10)
\[
\|\alpha^2 R_\alpha^2 A_s^{\frac{s}{2s+2a}} L^{-\frac{s}{2}} \hat{y}\| \leq \|\alpha^2 R_\alpha^2 A_s^{\frac{s}{2s+2a}} L^{-\frac{s}{2}} (I - P_h) \hat{y}\| + \|\alpha^2 R_\alpha^2 A_s^{\frac{s}{2s+2a}} L^{-\frac{s}{2}} P_h \hat{y}\| \\
\leq \hat{g}(s) \|\|I - P_h\|\| \|\hat{y}\| + F(\alpha, \hat{y}) \\
\leq \hat{g}(s) [\|I - P_h\| \|\hat{y} - y\|] + \|I - P_h\| A \|\hat{x}\| + F(\alpha, \hat{y}) \tag{5.16}
\]

Therefore if \(\alpha := \alpha(\delta, h)\) is the unique solution of (5.13), then we have
\[
\|\alpha^2 R_\alpha^2 A_s^{\frac{s}{2s+2a}} L^{-\frac{s}{2}} \hat{y}\| \leq (b + \hat{g}(s)) \delta + (d + \hat{g}(s) c_3 \|\hat{x}\|) h \tag{5.17}
\]

Now the result follows from (5.14), (5.15), (5.16) and (5.17), \(\Box\)

**Lemma 5.4** Suppose that \(\hat{x}\) belongs to \(H_t\) for some \(t \leq s\), and \(\alpha := \alpha(\delta, h) > 0\) is the unique solution of \(5.13\), where \(b > \hat{g}(s)\) and \(d > c_3 \|\hat{x}\| \hat{g}(s)\) with \(\hat{g}(s) := g\left(\frac{-s}{2s+2a}\right)\). Then
\[
\alpha \geq c_0 \delta^{\frac{s}{s+2a}}, \quad c_0 = \frac{\min\{b - \hat{g}(s), d - c_3 \|\hat{x}\| \hat{g}(s)\}}{g(\frac{-s}{2s+2a}) \rho} \tag{5.18}
\]

**Proof** Note that by (5.11), Proposition 2.1 and (2.6), we have
\[
F(\alpha, \hat{y}) \leq \alpha \|R_\alpha A_s^{\frac{s}{2s+2a}} L^{-\frac{s}{2}} P_h \hat{y}\| \\
\leq \alpha \|R_\alpha A_s^{\frac{s}{2s+2a}} L^{-\frac{s}{2}} P_h (\hat{y} - y)\| \\
+ \alpha \|R_\alpha A_s^{\frac{s}{2s+2a}} L^{-\frac{s}{2}} (P_h - I) y\| + \alpha \|R_\alpha A_s^{\frac{s}{2s+2a}} A_s L^2 \hat{x}\| \\
\leq \alpha \|R_\alpha A_s^{\frac{s}{2s+2a}} L^{-\frac{s}{2}} P_h (\hat{y} - y)\| \\
+ \alpha \|R_\alpha A_s^{\frac{s}{2s+2a}} L^{-\frac{s}{2}} (P_h - I) y\| + \alpha \|R_\alpha A_s^{\frac{s}{2s+2a}} A_s L^2 \hat{x}\| \\
\leq \alpha \|R_\alpha A_s^{\frac{s}{2s+2a}} L^{-\frac{s}{2}} P_h (\hat{y} - y)\| \\
+ \alpha \|R_\alpha A_s^{\frac{s}{2s+2a}} L^{-\frac{s}{2}} (P_h - I) y\| + \alpha \|R_\alpha A_s^{\frac{s}{2s+2a}} A_s^2 L^2 \hat{x}\| \\
\leq \hat{g}(s) [\delta + c_3 \|\hat{x}\| h] + \|\alpha R_\alpha A_s^{\frac{s}{2s+2a}} A_s L^2 \hat{x}\| \\
\leq \hat{g}(s) [\delta + c_3 \|\hat{x}\| h] + g \left(\frac{s - 2t}{2s + 2a}\right) \rho \alpha^{\frac{s}{s+2a}}.
\]
Thus
\[
\min\{b - \tilde{g}(s), d - c_3 \|\hat{x}\|\tilde{g}(s)\}\{(\delta + h)\leq g\left(\frac{s - 2t}{2s + 2a}\right)\rho a^{\frac{\tau}{1+\alpha}}
\]
which implies
\[
\alpha \geq c_0(\delta + h)^{\frac{\tau}{1+\alpha}}, \quad c_0 = \min\{b - \tilde{g}(s), d - c_3 \|\hat{x}\|\tilde{g}(s)\}\frac{g\left(\frac{s - 2t}{2s + 2a}\right)\rho}{\alpha}
\]
This completes the proof. \[
\]
**THEOREM 5.5** Under the assumptions in Lemma 5.4, for any fixed \(\tau > 0\),
\[
\|\hat{x}_\alpha - \hat{x}_{\alpha,h}\| \leq c_5(\delta + h)^{\zeta},
\]
where\[
\zeta := \frac{\tau}{\tau + 1} + \frac{s}{2} - \frac{s + 2a}{2t + 2a} + \gamma, \quad c_5 \geq c_4 f\left(\frac{1}{2}\right)^{-\frac{\tau}{1+\alpha}} \max\{g\left(\frac{1}{2}\right), 1\}
\]
with\[
\gamma := \begin{cases} 
0 & \text{if } t \geq 1 - a \\
\frac{s + a}{2} \left(1 - \frac{1}{t+\alpha}\right) & \text{if } t < 1 - a.
\end{cases}
\]
**Proof.** Note that by (3.9), Lemme 5.4 and Lemma 5.3,
\[
\|\hat{x}_\alpha - \hat{x}_{\alpha,h}\| \leq f\left(\frac{1}{2}\right)^{-\frac{\tau}{1+\alpha}} \Phi(s, h, \alpha)\alpha^{-\frac{s + a}{1+\alpha}} h^s \|\hat{x}_\alpha\|,
\]
\[
\leq cf\left(\frac{1}{2}\right)^{-\frac{\tau}{1+\alpha}} \max\{g\left(\frac{1}{2}\right)h^{\frac{s}{2} - \frac{1}{2}}, \alpha^{\frac{1}{2}h^{\frac{1}{2}} - \frac{s - 1}{2}}\} c_4 \alpha^{-\frac{s + a}{1+\alpha}} h^s \alpha^{-1}(\delta + h)^{\frac{\tau}{1+\alpha}}
\]
\[
\leq cf\left(\frac{1}{2}\right)^{-\frac{\tau}{1+\alpha}} \max\{g\left(\frac{1}{2}\right)h^{\frac{s + a}{2} - \frac{1}{2}}, \alpha^{\frac{s + a}{2} - \frac{1}{2}}, 1\} c_4 \alpha^{-\frac{s + a}{1+\alpha}} h^s (\delta + h)^{\frac{\tau}{1+\alpha}}
\]
\[
\leq cf\left(\frac{1}{2}\right)^{-\frac{\tau}{1+\alpha}} \max\{g\left(\frac{1}{2}\right)(\delta + h)^{\frac{s + a}{2} - \frac{1}{2}}, 1\} \times
\]
\[
c_4(\delta + h)^{\frac{s + a}{2} - \frac{1}{2} - \frac{s + a}{1+\alpha}} \alpha^{\frac{s + a}{2} - \frac{1}{2}} + \gamma.
\]
This completes the proof. \[
\]
**THEOREM 5.6** Under the assumptions in Lemma 5.4,
\[
\|\hat{x} - x_\alpha\| = O((\delta + h)^{\frac{\tau}{1+\alpha}}).
\]
Proof. Since \( x_\alpha \) is the solution of (2.9), we have
\[
\hat{x} - x_\alpha = \hat{x} - (A + \alpha L^*)^{-1} y \\
= \alpha L^{-\frac{z}{2}} (A\alpha + \alpha I)^{-1} L^\alpha \hat{x} \\
= \alpha L^{-\frac{z}{2}} R_\alpha L^\alpha \hat{x}.
\]
Therefore by (5.6), we have
\[
f\left( \frac{s}{2s + 2a} \right) \| \hat{x} - x_\alpha \| \leq \| \alpha A^{\frac{s-2a}{2s+2a}} R_\alpha L^\alpha \hat{x} \|. \tag{5.19}
\]
To obtain an estimate for \( \| \alpha A^{\frac{s-2a}{2s+2a}} R_\alpha L^\alpha \hat{x} \| \), first we shall make use the moment inequality (2.5) with
\[
u = \frac{t}{a}, \quad v = 1 + \frac{t}{a}, \quad B = \alpha R_\alpha A^{\frac{s}{2a}}, \quad x = \alpha^{-\frac{1}{2}} R_\alpha^{-\frac{1}{2}} A^{\frac{s-2t}{2s+2a}} L^\alpha \hat{x}.
\]
Then since
\[
\| x \| \leq \| A^{\frac{s-2t}{2s+2a}} L^\alpha \hat{x} \| \leq g\left( \frac{s - 2t}{2a} \right) \| L^\alpha \hat{x} \|_{\frac{1}{1+\frac{1}{s}}} \leq g\left( \frac{s - 2t}{2s + 2a} \right) \rho
\]
we have
\[
\| \alpha A^{\frac{s-2a}{2s+2a}} R_\alpha L^\alpha \hat{x} \| = \| B^\frac{1}{\alpha} x \| \\
\leq \| B^{1+\frac{1}{\alpha}} x \| \| x \|^{\frac{1}{\alpha}} \\
\leq \| \alpha^2 R_\alpha^2 A^{\frac{s-2a}{2s+2a}} L^\alpha \hat{x} \| \| x \|^{\frac{1}{\alpha}} \\
\leq \| \alpha^2 R_\alpha^2 A^{\frac{s-2a}{2s+2a}} L^\alpha \hat{x} \| \| x \|^{\frac{1}{\alpha}} \\
\leq g\left( \frac{s - 2t}{2s + 2a} \right) \| x \|^\frac{1}{\alpha} \| \alpha^2 R_\alpha^2 A^{\frac{s-2a}{2s+2a}} L^\alpha \hat{x} \|^{\frac{1}{\alpha}}. \tag{5.20}
\]
Further by (5.2), (5.6) and (5.10),
\[
\| \alpha^2 R_\alpha^2 A^{\frac{s-2a}{2s+2a}} L^\alpha \hat{x} \| \leq \| \alpha^2 R_\alpha^2 A^{\frac{s-2a}{2s+2a}} L^\alpha \hat{x} \| \| y - \hat{y} \| + \| \alpha^2 R_\alpha^2 A^{\frac{s-2a}{2s+2a}} L^\alpha \hat{x} \| (I - P_h) \hat{y} \|
\]
\[
+ \| \alpha^2 R_\alpha^2 A^{\frac{s-2a}{2s+2a}} L^\alpha \hat{x} \| P_h \hat{y} \|
\]
\[
\leq g\left( \frac{s}{2s + 2a} \right) (\delta + c_3 \| \hat{x} \| h) + F(\alpha, \hat{y}) \tag{5.21}
\]
Therefore if \( \alpha = \alpha(\delta, h) \) is the unique solution of (5.13), then we have
\[
\| \alpha^2 R_\alpha^2 A^{\frac{s-2a}{2s+2a}} L^\alpha \hat{x} \| \leq \| \hat{y}(s) + b \| \delta + [\hat{y}(s)] c_3 \| \hat{x} \| + d \| h \|. \tag{5.22}
\]
Now the result follows from (5.19), (5.20), (5.21) and (5.22). \( \Box \)
THEOREM 5.7 Under the assumptions in Lemma 5.4, for any fixed \( \tau > 0 \),
\[
\| \hat{x} - \tilde{x}_{\alpha,h} \| \leq c_6 (\delta + h)^\mu, \quad \mu := \min \left\{ \frac{t}{t + a}, \frac{\tau}{\tau + 1} \right\}
\]
for some \( c_6 > 0 \), and \( \zeta \) as in Theorem 5.5.

Proof. Let \( x_\alpha \) and \( \hat{x}_\alpha \) be the solutions of (2.9) and (2.10) respectively. Then by triangle inequality, (5.3) and Proposition 2.1,
\[
\| \hat{x} - \tilde{x}_{\alpha,h} \| \leq \| \hat{x} - x_\alpha \| + \| x_\alpha - \hat{x}_\alpha \| + \| \hat{x}_\alpha - \tilde{x}_{\alpha,h} \|
\]
\[
= \| \hat{x} - x_\alpha \| + \frac{1}{f(\frac{s}{2s + 2a})} \| A_s^{\frac{s}{2s + 2a}} R_\alpha L^{-\frac{s}{2}} (y - \tilde{y}) \| + \| \hat{x}_\alpha - \tilde{x}_{\alpha,h} \|
\]
\[
\leq \| \hat{x} - x_\alpha \| + \frac{1}{f(\frac{s}{2s + 2a})} \| A_s^{\frac{s}{2s + 2a}} R_\alpha A_s^{\frac{s}{2s + 2a}} L^{-\frac{s}{2}} (y - \tilde{y}) \| + \| \hat{x}_\alpha - \tilde{x}_{\alpha,h} \|
\]
\[
\leq \| \hat{x} - x_\alpha \| + \frac{1}{f(\frac{s}{2s + 2a})} \| A_s^{\frac{s}{2s + 2a}} R_\alpha \| \| A_s^{\frac{s}{2s + 2a}} L^{-\frac{s}{2}} (y - \tilde{y}) \| + \| \hat{x}_\alpha - \tilde{x}_{\alpha,h} \|
\]
\[
\leq \| \hat{x} - x_\alpha \| + \frac{g(-\frac{s}{2s + 2a})}{f(\frac{s}{2s + 2a})} (\delta a - \frac{s}{2s + 2a}) + \| \hat{x}_\alpha - \tilde{x}_{\alpha,h} \|
\]
\[
\leq \| \hat{x} - x_\alpha \| + \frac{g(-\frac{s}{2s + 2a})}{f(\frac{s}{2s + 2a})} (\delta + h) a - \frac{s}{2s + 2a} + \| \hat{x}_\alpha - \tilde{x}_{\alpha,h} \| \quad (5.23)
\]
The proof now follows from Lemma 5.4, Theorem 5.5 and Theorem 5.6. \( \square \)

COROLLARY 5.8 If \( t, s, a \) satisfy \( \max\{0, 1 - a\} < t \leq s \) and \( \tau \) is large enough such that
\[
\gamma + \frac{s}{2} \left[ 1 - \frac{1}{t + a} \right] \geq \frac{1}{\tau + 1},
\]
then
\[
\| \hat{x} - \tilde{x}_{\alpha,h} \| \leq c_6 (\delta + h)^{\frac{1}{\tau + 1}}
\]
with \( c_6 \) as in Theorem 5.7.

Proof. Let \( \zeta, \mu \) be as in Theorems 5.5 and 5.7 respectively. Then we observe that
\[
\mu = \frac{t}{t + a} \quad \text{if} \quad \gamma + \frac{s}{2} \left[ 1 - \frac{1}{t + a} \right] \geq \frac{1}{\tau + 1}.
\]
Hence the result follows from Theorem 5.7. \( \square \)
6 Order Optimality of the Error Estimates

In order to measure the quality of an algorithm to solve an equation of the form (2.1), Miccelli and Rivlin [11] considered the quantity

\[ e(M, \delta) := \sup\{\|x\| : x \in M, \|Ax\| \leq \delta\} \]

and showed that

\[ e(M, \delta) \leq E(M, \delta) \leq 2e(M, \delta), \tag{6.1} \]

where

\[ E(M, \delta) = \inf_R \sup\{\|x - Rv\| : x \in M, v \in H, \|Ax - v\| \leq \delta\} \]

is the best possible worst error. Here the stabilizing set \( M \) is assumed to be convex such that \( M = -M \) with \( 0 \in M \) (see also, Vainikko and Varetenikov [22]), and infimum is taken over all algorithms \( R : Y \rightarrow X \). Since \( H \) is a Hilbert space and \( A \) is assumed to be self-adjoint and positive we, in fact, have (c.f. Melkman and Miccelli [10])

\[ e(M, \delta) = E(M, \delta). \]

Now using the assumption (2.6), and taking \( r = -a, \lambda = 0 \) in the interpolation inequality (2.4), we obtain

\[ \|x\| \leq \|x\|^{t/(t+a)} \|x\|_t^{a/(t+a)} \leq \left( \|Ax\|/c_1 \right)^{t/(t+a)} \|x\|_t^{a/(t+a)}, \quad x \in H_t. \]

Therefore for the set

\[ M_{t, \rho} = \{x : \|x\|_t \leq \rho\} \]

with a fixed \( t > 0, \rho > 0 \),

\[ e(M_{t, \rho}, \delta) \leq \left( \delta/c_1 \right)^{t/(t+a)} \rho^{a/(t+a)}. \]

It is known that the above estimate for \( e(M_{t, \rho}, \delta) \) is sharp (c.f. Vainikko [21]). In view of the above observations, an algorithm is called an optimal order yielding algorithm with respect to \( M_{t, \rho} \) and the assumption (2.6), if it yields an approximation \( \hat{x} \) corresponding to the data \( \hat{y} \) with \( \|y - \hat{y}\| \leq \delta \) satisfying

\[ \|\hat{x} - \hat{x}\| = O(\delta^{t/(t+a)}), \quad x \in H_t. \]

Clearly Corollary 5.8 shows that if \( h = O(\delta) \) and if \( \max\{0, 1 - a\} < t \leq s \) and \( \tau \) is large enough such that

\[ \gamma + \frac{s}{2} \left[ 1 - \frac{1}{t + a} \right] \geq \frac{1}{\tau + 1}, \]

then we obtain the optimal order.
7 Applications

For $r \geq 2$, denote by $S_h$ the space of $r$th order splines on the uniform mesh of width $h = \frac{1}{n}$, that is, $S_h$ consists of functions in $C^{r-1}[0,1]$ which are piecewise polynomials of degree $r - 2$. For positive integers $s$, let $H^s$ denote the Sobolev space of functions $u \in C^{s-1}[0,1]$ with $u^{s-1}$ absolutely continuous and the norm $\|u\|_{H^s}$ defined by

$$\|u\|_{H^s} = \left(\sum_{i=1}^{s} \|u^{(i)}\|^2\right)^{\frac{1}{2}}, \quad u \in H^s.$$  

Then $S_h$ is a finite dimensional subspace of $H^{r-1}$ which has the following well-known approximation property (cf. [1]): For $u \in H^s$, $s \in \mathbb{N}$, there is a constant $\kappa$ (independent of $h$) such that

$$\inf_{\varphi \in S_h} \|u - \varphi\|_{H^j} \leq \kappa h^{\min\{s, r\} - j} \|u\|_{H^s}, \quad u \in H^s, \quad j \in \{0, 1\}$$  \hspace{1cm} (7.1)

so that assumption (2.15) is satisfied. We take $L$ is as in (2.7), i.e.,

$$Lx := \sum_{j=1}^{\infty} j^2 \langle x, u_j \rangle u_j,$$  \hspace{1cm} (7.2)

where $u_j(t) := \sqrt{2} \sin(j\pi t)$, $j \in \mathbb{N}$ with domain of $L$ as

$$D(L) := \{ x \in L^2[0,1] : \sum_{j=1}^{\infty} j^4 |\langle x, u_j \rangle|^2 < \infty \}.$$  

In this case $(H_t)_{t \in \mathbb{R}}$ is given as in (2.8). It can be seen that

$$H_t = \{ x \in L^2[0,1] : \sum_{j=1}^{\infty} j^{4l} |\langle x, u_j \rangle|^2 < \infty \}$$  

$$= \{ x \in H^{2l}(0,1) : x^{(2l)}(0) = x^{(2l)}(1) = 0, l = 0, 1, \ldots, \left\lfloor \frac{t - 1}{4} \right\rfloor \},$$

where $\left\lfloor p \right\rfloor$ denotes the greatest integer less than or equal to $p$. We observe that $H^0 = L^2[0,1]$, and for $s \in \mathbb{N}$, $H_s \subset H^s$.

Now, let $A : L^2[0,1] \to L^2[0,1]$ be a positive self adjoint operator. Then we have

$$\|A(I - P_h)\| = \|(I - P)A\| = \sup_{\|u\| \leq 1} \inf_{\varphi \in S_h} \|Au - \varphi\|.$$  

Hence by (7.1)

$$\|A(I - P_h)\| \leq \kappa h^{\min\{s, r\}} \sup_{\|u\| \leq 1} \|Au\|_{H^s}.$$
From the above inequality it is clear that if $Au \in H_s$ for every $u \in L^2[0,1]$, and if $A : L^2[0,1] \to H_s$ is a bounded operator, then there exists a constant $\hat{c}$ such that

$$\|A(I - P_h)\| \leq \kappa \hat{c} h^{\min(s,r)} \quad (7.3)$$

so that (5.2) is satisfied.

Now, let us consider the case of an integral operator, namely, (2.2):

$$(Ax)(\xi) = \int_0^1 k(\xi,t)x(t) \, dt, \quad 0 \leq \xi \leq 1,$$

having all its eigenvalues non-negative, and $k(\xi,t) = k(t,\xi)$ for all $(\xi,t) \in [0,1] \times [0,1]$ is such that it is differentiable $s$ times with respect to the variable $\xi$ with its $s$-th derivative lies in $L^2[0,1]$. For example, the integral operator may be the one associated with the \textit{backward heat equation} problem considered in Section 2.

Now,

$$\frac{d^s}{d\xi^s}(Au)(\xi) = \int_0^1 \frac{\partial^s k(\xi,t)}{\partial \xi^s} u(t) \, dt$$

so that

$$\|Au\|_{H^s} \leq \|k\|_{0,s}\|u\| \quad \text{with} \quad \|k\|_{0,s} = \sum_{i=0}^s \int_0^1 \int_0^1 \left| \frac{\partial^i k(t,\xi)}{\partial \xi^i} \right|^2 \, dt \, d\xi.$$

Thus we get (7.3) with $\hat{c} = \|k\|_{0,s}$.

\section*{References}


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The following corrections have been incorporated (in the original version of the paper) which includes all the corrections suggested by the referee. The page reference is as in the original version which the referee had. line (+n) and line (+n) denotes line n from above and line n from below respectively - section headings are excluded from counting.

- In the abstract: the last but one sentence (“Algorithm ... were given”) is omitted.
- “noise” has been changed to “noise” throughout the text.
- Introduction, Page 1, line (+3): “which often occur” changed to “that often occurs”
- Introduction, Page 1, line (+3): In section of Preliminaries: The sentence “Here and in what follows ⟨·,·⟩ denotes the inner product on H with the corresponding norm ||·||.” is incorporated as second sentence.
- Page 3: line (+3): “finial” changed to “final”
- Page 3: line (+13): “the above” changed to “The above”
- Page 3: line (+14): “sin(nξ)” changed to “sin(nπξ)”
- Page 3: line (-1): “with” changed to “and”
- Page 4: line (-4): “of integral” changed to “of the integral”
- Page 6: line (-7): “(2.15,” changed to “(2.15),”
- Page 8: line (-7): “−u” changed to “−ϕ”
- Page 10: first line of Theorem 4.1: “0 < t < s + a” changed to “0 < t ≤ s + a”
- Pages 12-14: “d” changed to “∂”
- Pages 13: first line after equation (5.7): “where and” changed to “where”
- Pages 18: End of the proof of Theorem 5.5: “This completes the proof” changed to “This completes the proof.”
- Pages 19: End of the the statement of Theorem 5.7: “c6 with ζ is” changed to “c6 > 0 and ζ”
- Pages 20: First sentence of the proof of Theorem 5.7: “and Proposition 2.2;” changed to “and Proposition 2.1,”
- Pages 20: First sentence of Corollary 5.8 omitted.
- Pages 20: In the proof of Corollary 5.8: “We observe that” changed to “Let ζ and µ be as in Theorem 5.5. and 5.7 respectively. Then”
- Pages 20: Last sentence in the proof of Corollary 5.8: “the above theorem.” changed to “Theorem 5.7.”
• Pages 22: In the section applications:
  “Here ‖·‖ denotes $L^2$-norm.” is removed.
• Page 22: line (-4):
  “integer or equal to” changed to “integer less than or equal to”
• Page 23: line (+11): “withe” changed to “with”
• Page 23: line (+12): “Section 1.” changed to “Section 2.”
• Page 23: line (+17): “from (7.3),” omitted.
• Page 24: in reference [9], “P.M.Petunin” changed to “P.I.J.Petunin”
• Some modalities (such as slant/italics ect.) of writing the references have been corrected.