AN OPTIMAL ORDER YIELDING DISCREPANCY PRINCIPLE FOR SIMPLIFIED REGULARIZATION OF ILL-POSED PROBLEMS IN HILBERT SCALES

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ABSTRACT

Recently, Tautenhahn and Hämärik (1999) have considered a monotone rule as a parameter choice strategy for choosing the regularization parameter while considering approximate solution of an ill-posed operator equation $Tx = y$, where $T$ is a bounded linear operator between Hilbert spaces. Motivated by this, we propose a new discrepancy principle for the simplified regularization, in the setting of Hilbert scales, when $T$ is a positive and self adjoint operator. When the data $y$ is known only approximately, our method provides optimal order under certain natural assumptions on the ill-posedness of the equation and smoothness of the solution. The result, in fact, improves an earlier work of the authors (1997)

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1. Introduction

Tikhonov regularization (cf. [2]) is one of the most widely used procedure to obtain stable approximate solution to an ill-posed operator equation

$$Tx = y,$$

where $T : X \to Y$ is a bounded linear operator between Hilbert spaces $X$ and $Y$. Suppose the data $y$ is not known exactly, but only an approximation of it, namely $\tilde{y}$, is available. Then, the regularized solution $\tilde{x}_\alpha$, by Tikhonov regularization, is obtained by minimizing the map

$$x \mapsto \|Tx - \tilde{y}\|^2 + \alpha\|x\|^2$$

for $\alpha > 0$. For $y \in R(T) + R(T)^\bot$, if $\hat{x}$ is the generalized solution of (1.1), i.e., $\hat{x} = T^\dagger y$, where $T^\dagger$ is the Moore-Penrose generalized inverse of $T$, then estimates for the error $\|\hat{x} - \tilde{x}_\alpha\|$ are obtained by choosing the regularization parameter $\alpha$ appropriately. It is known that (see, for example [2]), if $\hat{x} \in R((T^*T)^\nu)$ for some $\nu > 0$, and if $\|y - \tilde{y}\| \leq \delta$ for some noise level $\delta > 0$, then the optimal order for the above error is $O(\delta^\mu)$, where $\mu = \min\{2\nu/(2\nu + 1), 2/3\}$.

In order to improve the error estimates available in Tikhonov regularization, Natterer [9] carried out error analysis in the frame work of Hilbert scales. Subsequently, many authors extended, modified and generalized Natterer’s work to obtain error
bounds under various contexts (see, e.g., Natterer [9], Hegland [3], Schröter and Tautenhahn [12], Mair [6], Nair, Hegland and Anderssen [8], Nair [7]).

If $T$ is a positive, self-adjoint operator on a Hilbert space, then the simplified regularization introduced by Lavrentiev is better suited than Tikhonov regularization in terms of speed of convergence and condition number in the case of finite dimensional approximations (cf. Schock [11]).

In [1], simplified regularization in the framework of Hilbert scales was studied for the first time, and obtained error estimates under a priori and a posteriori parameter choice strategies. The a posteriori choice of the parameter in that paper has a drawback that it can yield the optimal rate only under certain restricted smoothness assumption on the solution.

In this paper we propose a new discrepancy principle, for choosing the regularization parameter $\alpha$, for simplified regularization in the setting of Hilbert scales, which eliminates the drawback of the method in [1], yielding the optimal order for a range of values of smoothness. The discrepancy principle of this paper is motivated by a recent procedure adopted by Tautenhahn and Hämmerik [13].

2. Preliminaries

Let $H$ be a Hilbert space and $A : H \to H$ be a bounded, positive, self-adjoint operator on $H$. Recall that $A$ is said to be a positive operator if $\langle Ax, x \rangle \geq 0$ for every $x \in H$. For $y \in R(A)$, the range of $A$, consider the operator equation

$$Ax = y,$$

(2.2)

Let $\hat{x}$ be the minimal norm solution of (2.2). It is well-known that if $R(A)$ is not closed in $H$, then the problem of solving (2.2) for $\hat{x}$ is ill-posed, in the sense that small perturbation in the data $y$ can cause large deviations in the solution.

A prototype of the equation (2.2) is an integral equation of the first kind,

$$\int_0^1 k(s,t)x(t)\,dt = y(s), \quad 0 \leq s \leq 1,$$

where $k(\cdot, \cdot)$ is a non-degenerate kernel which is square integrable, i.e.,

$$\int_0^1 \int_0^1 |k(s,t)|^2 \,dt\,ds < \infty,$$

satisfying $k(s,t) = k(t,s)$ for all $s,t$ in $[0, 1]$, and such that the eigenvalues of the corresponding integral operator $A : L^2[0, 1] \to L^2[0, 1]$,

$$(Ax)(s) = \int_0^1 k(s,t)x(t)\,dt, \quad 0 \leq s \leq 1,$$

(2.3)

are all non-negative. For example, consider the kernel $k(\cdot, \cdot)$ defined by

$$k(s,t) = \begin{cases} 
(1 - s)t & \text{if } 0 \leq s \leq t \leq 1, \\
(1 - t)s & \text{if } 0 \leq t \leq s \leq 1,
\end{cases}$$
Clearly, $k(s, t) = k(t, s)$, so that $A : L^2[0, 1] \to L^2[0, 1]$ defined as in (2.3) is a self-adjoint operator. Moreover, the eigenvalues of this operator are $1/n^2 \pi^2$ for $n = 1, 2, \ldots$ (see Limaye [5], Page 329).

For considering the regularization of the equation (2.2) in the setting of Hilbert scales, we consider a Hilbert scale $\{H_t\}_{t \in \mathbb{R}}$ generated by a strictly positive operator $L : D(L) \to H$ with its domain $D(L)$ dense in $H$ satisfying

$$\|Lx\| \geq \|x\|, \quad x \in D(L).$$

Recall (cf. [4]) that the space $H_t$ is the completion of $D := \bigcap_{k=0}^{\infty} D(L^k)$ with respect to the norm $\|x\|_t$, induced by the inner product

$$\langle u, v \rangle_t = \langle L^t u, L^t v \rangle, \quad u, v \in D.$$

Moreover, if $\beta \leq \gamma$, then the embedding $H_\gamma \hookrightarrow H_\beta$ is continuous, and therefore the norm $\|x\|_\beta$ is also defined in $H_\gamma$ and there is a constant $c_{0,1}$ such that

$$\|x\|_\beta \leq c_{0,1}\|x\|_\gamma, \quad x \in H_\gamma. \quad (2.4)$$

We assume that the ill-posed nature of the operator $A$ is related to the Hilbert scale $\{H_t\}_{t \in \mathbb{R}}$ according to the relation

$$c_1\|x\|_{-a} \leq \|Ax\| \leq c_2\|x\|_{-a}, \quad x \in H, \quad (2.5)$$

for some positive reals $a$, $c_1$ and $c_2$.

For the example of an integral operator given in the previous paragraph, one may take $L$ to be defined by

$$Lx := \sum_{j=1}^{\infty} j^2 \langle x, u_j \rangle u_j,$$

where $u_j(t) := \sqrt{2} \sin(j \pi t)$, $j \in \mathbb{N}$, and domain of $L$ is

$$D(L) := \{x \in L^2[0, 1] : \sum_{j=1}^{\infty} j^4 |\langle x, u_j \rangle|^2 < \infty\}.$$ 

In this case, it can be seen that

$$H_t = \{x \in L^2[0, 1] : \sum_{j=1}^{\infty} j^{4t} |\langle x, u_j \rangle|^2 < \infty\}$$

and the constants $a$, $c_1$ and $c_2$ in (2.5) are given by $a = 1$, $c_1 = c_2 = 1/\pi^2$ (see Schröter and Tautenhahn [12], Section 4).

As in [1], we consider the regularized solution of (1.1) as the solution of the well-posed equation

$$(A + \alpha L^s)x_{\alpha} = y, \quad \alpha > 0, \quad (2.6)$$

where $s$ is a fixed non-negative real number.
Suppose the data \( y \neq 0 \) is known only approximately, say \( \tilde{y} \neq 0 \) with \( \| y - \tilde{y} \| \leq \delta \) for a known error level \( \delta > 0 \). Then, in place of (2.6), we consider

\[
(A + \alpha L^s)\tilde{x}_\alpha = \tilde{y}.
\]

(2.7)

It can be seen that the solution \( \tilde{x}_\alpha \) of the above equation is the unique minimizer of the function

\[
x \mapsto \langle Ax, x \rangle - 2\langle \tilde{y}, x \rangle + \alpha \langle L^s x, x \rangle, \quad x \in D(L).
\]

(2.8)

We also observe that, taking \( A_s := L^{-s/2}AL^{-s/2} \), the equations (2.6) and (2.7) take the forms

\[
L^{s/2}(A_s + \alpha I)L^{s/2}x_\alpha = y, \quad L^{s/2}(A_s + \alpha I)L^{s/2}\tilde{x}_\alpha = \tilde{y}
\]

respectively. Note that the operator \( A_s \) defined above is positive and self adjoint bounded operator on \( H \).

One of the crucial results for proving the results in [1] as well as the results in this paper is the following result, where functions \( f \) and \( g \) are defined by

\[
f(t) = \min\{c_1 t, c_2 t\}, \quad g(t) = \max\{c_1 t, c_2 t\}, \quad t \in \mathbb{R}, \quad |t| \leq 1,
\]

(2.9)

PROPOSITION 2.1. (Proposition 3.1 [1]) For \( s \geq 0 \) and \( |\nu| \leq 1 \),

\[
f\left(\frac{\nu}{2}\right) \| x\|_{\nu^{(s+a)}} \leq \| A_s^\frac{s}{2}x\| \leq g\left(\frac{\nu}{2}\right) \| x\|_{\nu^{(s+a)}}, \quad x \in H.
\]

Using the above proposition the following result has been proved in George and Nair [1].

THEOREM 2.2. (Theorem 3.2 [1]) Suppose \( \hat{x} \in H_t, 0 < t \leq s + a \) and \( \alpha > 0 \). Then

\[
\| \hat{x} - \tilde{x}_\alpha \| \leq \phi(s, t)\alpha^{\frac{s}{(s+a)}} \| x\|_t + \psi(s)\alpha^{-\frac{s}{(s+a)}} \delta,
\]

(2.10)

where

\[
\phi(s, t) = \frac{g\left(\frac{s-2t}{2s+2a}\right)}{f\left(\frac{s}{2s+2a}\right)}, \quad \psi(s) = \frac{g\left(-\frac{s}{2s+2a}\right)}{f\left(\frac{s}{2s+2a}\right)}.
\]

In particular, if \( \alpha = c_0 \delta^{\frac{1}{(s+a)}} \) for some constant \( c_0 > 0 \), then

\[
\| \hat{x} - \tilde{x}_\alpha \| \leq \eta(s, t)\delta^{\frac{1}{(s+a)}}
\]

where

\[
\eta(s, t) = \max\{\phi(s, t)\| \hat{x}\|_{c_0^{\frac{1}{(s+a)}}}, \psi(s)c_0^{-\frac{1}{(s+a)}}\}.
\]
Let \( R_\alpha = (A_s + \alpha I)^{-1} \). We shall make use of the relation
\[
\|R_\alpha A_s^\tau\| \leq \alpha^{\tau-1}, \quad \alpha > 0; \ 0 < \tau \leq 1,
\]
which follows from the spectral properties of the self-adjoint operator \( A_s, s > 0 \).

In [1], the authors considered parameter choice strategies, a priori and a posteriori, which yield the optimal rate \( O(\delta t^t + a) \) if \( \hat{x} \in H_t \) for certain specific values of \( t \). The a posteriori parameter choice strategy in [1] is to choose \( \alpha \) such that
\[
\alpha^{p+1}\|(A_s + \alpha I)^{-p-1}L^{-s/2}x\| = k\delta,
\]
where \( k > 1 \) and \( \tilde{y} \in X \) satisfy \( 0 < k\delta \leq \|\tilde{y}\|_{-s/2} \). Under the above procedure, the optimal order \( O\left(\delta^{1+t}\right) \) is obtained for \( t = s + p(s + a) \).

In the present paper, we propose a new discrepancy for choosing the regularization parameter \( \alpha \) which yield the optimal rate:
\[
\|\hat{x} - \tilde{x}_\alpha\| = O\left(\delta^{1+t}\right)
\]

3. THE DISCREPANCY PRINCIPLE

Let \( s, a \) be fixed positive real numbers. For \( \alpha > 0 \) and nonzero \( x \in H \), let
\[
\Phi(\alpha, x) := \frac{\alpha\|R_\alpha^3 A_s^{-s/2} L^{-2} x\|^2}{\|R_\alpha^2 A_s^{-s/2} L^{-2} x\|}.
\]
Note that, by the assumption (2.5), \( \|R_\alpha^2 A_s^{-s/2} L^{-2} x\| \) is nonzero for every nonzero \( x \in H \) so that the function \( \Phi(\alpha, x) \) is well-defined for every \( \alpha > 0 \) and for every nonzero \( x \in H \).

We assume that the available data \( \tilde{y} \) is nonzero and
\[
\|y - \tilde{y}\| \leq \delta
\]
for some known error level \( \delta > 0 \). Our idea is to prove the existence of a unique \( \alpha \) such that
\[
\Phi(\alpha, \tilde{y}) = c\delta
\]
for some known \( c > 0 \).

In the due course we shall make use of the relation
\[
f\left(\frac{-s}{2s+2a}\right)\|x\| \leq \|A_s^{-s/2} L^{-2} x\| \leq g\left(\frac{-s}{2s+2a}\right)\|x\|
\]
which can be easily derived from Proposition 2.1.

First we prove the monotonicity of the function \( \Phi(\alpha, x) \) defined in (3.12).
THEOREM 3.1. For each nonzero \( x \in H \), the function \( \alpha \mapsto \Phi(\alpha, x) \) for \( \alpha > 0 \), defined in (3.12), is increasing and it is continuously differentiable with \( \Phi'(\alpha, x) \geq 0 \). In addition

\[
\lim_{\alpha \to 0} \Phi(\alpha, x) = 0, \quad \lim_{\alpha \to \infty} \Phi(\alpha, x) = \| A_{s}^{\frac{\alpha}{s+2\alpha}} L \frac{x}{x} \|.
\]

Proof. Using (3.12) one can write

\[
\frac{d}{d\alpha}\Phi(\alpha, x) = \frac{d}{d\alpha}(\Phi^{2}(\alpha, x)) = 2\alpha \| R_{s}^{3/2} A_{s}^{\frac{\alpha}{s+2\alpha}} L \frac{x}{x} \|^{2} \| R_{s}^{3/2} A_{s}^{\frac{\alpha}{s+2\alpha}} L \frac{x}{x} \|^{2} \frac{d}{d\alpha}\left[ \alpha \| R_{s}^{3/2} A_{s}^{\frac{\alpha}{s+2\alpha}} L \frac{x}{x} \|^{2} \right]\\
- \frac{2\alpha\| R_{s}^{3/2} A_{s}^{\frac{\alpha}{s+2\alpha}} L \frac{x}{x} \|^{{2}}\| R_{s}^{3/2} A_{s}^{\frac{\alpha}{s+2\alpha}} L \frac{x}{x} \|^{{3}}}{\| R_{s}^{3/2} A_{s}^{\frac{\alpha}{s+2\alpha}} L \frac{x}{x} \|^{{3}}} - \frac{\alpha^{2}\| R_{s}^{3/2} A_{s}^{\frac{\alpha}{s+2\alpha}} L \frac{x}{x} \|^{{4}}\frac{d}{d\alpha}\left[ \| R_{s}^{3/2} A_{s}^{\frac{\alpha}{s+2\alpha}} L \frac{x}{x} \|^{{2}} \right]}{\| R_{s}^{3/2} A_{s}^{\frac{\alpha}{s+2\alpha}} L \frac{x}{x} \|^{{3}}}
\]

Thus,

\[
\frac{d}{d\alpha}\Phi(\alpha, x) = \frac{\| R_{s}^{3/2} A_{s}^{\frac{\alpha}{s+2\alpha}} L \frac{x}{x} \|^{{2}}\frac{d}{d\alpha}\left[ \alpha \| R_{s}^{3/2} A_{s}^{\frac{\alpha}{s+2\alpha}} L \frac{x}{x} \|^{{2}} \right]}{\| R_{s}^{3/2} A_{s}^{\frac{\alpha}{s+2\alpha}} L \frac{x}{x} \|^{{3}}} - \frac{\alpha\| R_{s}^{3/2} A_{s}^{\frac{\alpha}{s+2\alpha}} L \frac{x}{x} \|^{{2}}\frac{d}{d\alpha}\left[ \| R_{s}^{3/2} A_{s}^{\frac{\alpha}{s+2\alpha}} L \frac{x}{x} \|^{{2}} \right]}{2\| R_{s}^{3/2} A_{s}^{\frac{\alpha}{s+2\alpha}} L \frac{x}{x} \|^{{3}}}
\]

(3.16)

Let \( \{ E_{\lambda} : 0 \leq \lambda \leq a \} \) be the spectral family of \( A_{s} \), where \( a = \| A_{s} \| \). Then

\[
\frac{d}{d\alpha}(\alpha\| R_{s}^{3/2} A_{s}^{\frac{\alpha}{s+2\alpha}} L \frac{x}{x} \|^{{2}}) = \frac{d}{d\alpha}\int_{0}^{a} \frac{\alpha}{\lambda^{3}+\lambda^{2}+\lambda+1} d(E_{\lambda} L \frac{x}{x}, L \frac{x}{x})
\]

\[
= \int_{0}^{a} \left[ \frac{1}{\lambda^{3}+\lambda^{2}+\lambda+1} - \frac{3\alpha}{\lambda^{3}+\lambda^{2}+\lambda+1} \right] d(E_{\lambda} L \frac{x}{x}, L \frac{x}{x})
\]

\[
= \| R_{s}^{3/2} A_{s}^{\frac{\alpha}{s+2\alpha}} L \frac{x}{x} \|^{{2}} - 3\alpha\| R_{s}^{3/2} A_{s}^{\frac{\alpha}{s+2\alpha}} L \frac{x}{x} \|^{{2}}.
\]

(3.17)

Similarly

\[
\frac{d}{d\alpha}(\| R_{s}^{3/2} A_{s}^{\frac{\alpha}{s+2\alpha}} L \frac{x}{x} \|) = -4\| R_{s}^{3/2} A_{s}^{\frac{\alpha}{s+2\alpha}} L \frac{x}{x} \|^{{2}}.
\]

(3.18)
Therefore from (3.16), by using (3.17) and (3.18), we get

\[
\frac{d}{d\alpha} \Phi(\alpha, x) = \frac{\|R_\alpha^2 A_{s}^{\frac{3}{\alpha} + 2\alpha} L^{-\frac{2}{\alpha}} x\|^2 \left[\|R_\alpha^3 A_{s}^{\frac{5}{\alpha} + 2\alpha} L^{-\frac{2}{\alpha}} x\|^2 - 3\alpha\|R_\alpha^2 A_{s}^{\frac{3}{\alpha} + 2\alpha} L^{-\frac{2}{\alpha}} x\|^2 \right]}{\|R_\alpha^2 A_{s}^{\frac{3}{\alpha} + 2\alpha} L^{-\frac{2}{\alpha}} x\|^3} + 2\alpha\|R_\alpha^3 A_{s}^{\frac{5}{\alpha} + 2\alpha} L^{-\frac{2}{\alpha}} x\|^2 \|R_\alpha^5 A_{s}^{\frac{7}{\alpha} + 2\alpha} L^{-\frac{2}{\alpha}} x\|^2}{\|R_\alpha^2 A_{s}^{\frac{3}{\alpha} + 2\alpha} L^{-\frac{2}{\alpha}} x\|^3}
\]

The above equation can be rewritten as

\[
\frac{d}{d\alpha} \Phi(\alpha, x) = \Psi_1(\alpha, x) + \Psi_2(\alpha, x),
\]

where

\[
\Psi_1(\alpha, x) = \frac{\|R_\alpha^2 A_{s}^{\frac{3}{\alpha} + 2\alpha} L^{-\frac{2}{\alpha}} x\|^2 \left[\|R_\alpha^3 A_{s}^{\frac{5}{\alpha} + 2\alpha} L^{-\frac{2}{\alpha}} x\|^2 - 3\alpha\|R_\alpha^2 A_{s}^{\frac{3}{\alpha} + 2\alpha} L^{-\frac{2}{\alpha}} x\|^2 \right]}{\|R_\alpha^2 A_{s}^{\frac{3}{\alpha} + 2\alpha} L^{-\frac{2}{\alpha}} x\|^3},
\]

\[
\Psi_2(\alpha, x) = \frac{2\alpha\|R_\alpha^3 A_{s}^{\frac{5}{\alpha} + 2\alpha} L^{-\frac{2}{\alpha}} x\|^2 \|R_\alpha^5 A_{s}^{\frac{7}{\alpha} + 2\alpha} L^{-\frac{2}{\alpha}} x\|^2}{\|R_\alpha^2 A_{s}^{\frac{3}{\alpha} + 2\alpha} L^{-\frac{2}{\alpha}} x\|^3}.
\]

Since

\[
\|R_\alpha^3 A_{s}^{\frac{5}{\alpha} + 2\alpha} L^{-\frac{2}{\alpha}} x\|^2 = \langle (A_\alpha + \alpha I)^{-3} A_{s}^{\frac{3}{\alpha} + 2\alpha} L^{-\frac{2}{\alpha}} x, A_{s}^{\frac{5}{\alpha} + 2\alpha} L^{-\frac{2}{\alpha}} x \rangle,
\]

\[
\|R_\alpha^2 A_{s}^{\frac{3}{\alpha} + 2\alpha} L^{-\frac{2}{\alpha}} x\|^2 = \langle (A_\alpha + \alpha I)^{-3} A_{s}^{\frac{3}{\alpha} + 2\alpha} L^{-\frac{2}{\alpha}} x, (A_\alpha + \alpha I)^{-1} A_{s}^{\frac{3}{\alpha} + 2\alpha} L^{-\frac{2}{\alpha}} x \rangle
\]

we have

\[
\|R_\alpha^3 A_{s}^{\frac{5}{\alpha} + 2\alpha} L^{-\frac{2}{\alpha}} x\|^2 - 3\alpha\|R_\alpha^2 A_{s}^{\frac{3}{\alpha} + 2\alpha} L^{-\frac{2}{\alpha}} x\|^2 = \|A_{s}^{\frac{5}{\alpha} + 2\alpha} R_\alpha^2 L^{-\frac{2}{\alpha}} x\|^2.
\]

Also,

\[
\|R_\alpha^2 A_{s}^{\frac{3}{\alpha} + 2\alpha} L^{-\frac{2}{\alpha}} x\|^4 = \left[\langle R_\alpha^2 A_{s}^{\frac{3}{\alpha} + 2\alpha} L^{-\frac{2}{\alpha}} x, R_\alpha^2 A_{s}^{\frac{3}{\alpha} + 2\alpha} L^{-\frac{2}{\alpha}} x \rangle \right]^2 = \left[\langle R_\alpha^3 A_{s}^{\frac{5}{\alpha} + 2\alpha} L^{-\frac{2}{\alpha}} x, R_\alpha^5 A_{s}^{\frac{7}{\alpha} + 2\alpha} L^{-\frac{2}{\alpha}} x \rangle \right]^2 \leq \|R_\alpha^3 A_{s}^{\frac{5}{\alpha} + 2\alpha} L^{-\frac{2}{\alpha}} x\|^2 \|R_\alpha^5 A_{s}^{\frac{7}{\alpha} + 2\alpha} L^{-\frac{2}{\alpha}} x\|^2.
\]

Hence

\[
\Psi_1(\alpha, x) \geq 0, \quad \Psi_2(\alpha, x) \geq 0
\]

so that

\[
\frac{d}{d\alpha}(\Phi(\alpha, x)) = \Psi_1(\alpha, x) + \Psi_2(\alpha, x) \geq 0.
\]
To prove the last part of the theorem we observe that,
\[
\alpha^2 \| R_\alpha^2 A_{\frac{n}{n+2}} L^\alpha x \| - \Phi(\alpha, x) = \frac{\alpha^2 \| R_\alpha^2 A_{\frac{n}{n+2}} L^\alpha x \|^2 - \alpha \| R_\alpha^2 A_{\frac{n}{n+2}} L^\alpha x \|^2}{\| R_\alpha^2 A_{\frac{n}{n+2}} L^\alpha x \|}.
\]
Since
\[
\alpha^2 \| R_\alpha^2 A_{\frac{n}{n+2}} L^\alpha x \|^2 = \alpha \langle R_\alpha^3 A_{\frac{n}{n+2}} L^\alpha x, \alpha R_\alpha A_{\frac{n}{n+2}} L^\alpha x \rangle,
\]
and since \( \alpha R_\alpha - I = A_s R_\alpha = R_\alpha A_s \), we have
\[
\alpha^2 \| R_\alpha^2 A_{\frac{n}{n+2}} L^\alpha x \| - \Phi(\alpha, x) = \frac{-\alpha \langle R_\alpha^3 A_{\frac{n}{n+2}} L^\alpha x, A_s R_\alpha A_{\frac{n}{n+2}} L^\alpha x \rangle}{\| R_\alpha^2 A_{\frac{n}{n+2}} L^\alpha x \|} \leq 0. \tag{3.19}
\]
Hence
\[
\Phi(\alpha, x) \geq \alpha^2 \| R_\alpha^2 A_{\frac{n}{n+2}} L^\alpha x \| \geq \frac{\alpha^2 \| A_{\frac{n}{n+2}} L^\alpha x \|}{\| A_s \| + \alpha}.
\]
Also, we have
\[
\Phi(\alpha, x) = \frac{\alpha \langle R_\alpha A_{\frac{n}{n+2}} L^\alpha x, R_\alpha^2 A_{\frac{n}{n+2}} L^\alpha x \rangle}{\| R_\alpha^2 A_{\frac{n}{n+2}} L^\alpha x \|} \leq \alpha \| R_\alpha A_{\frac{n}{n+2}} L^\alpha x \|. \tag{3.20}
\]
Hence
\[
\left( \frac{\alpha}{\| A_s \| + \alpha} \right)^2 \| A_{\frac{n}{n+2}} L^\alpha x \| \leq \Phi(\alpha, x) \leq \alpha \| R_\alpha A_{\frac{n}{n+2}} L^\alpha x \|.
\]
From this, it follows that
\[
\lim_{\alpha \to 0} \Phi(\alpha, x) = 0 \quad \text{and} \quad \lim_{\alpha \to \infty} \Phi(\alpha, x) = \| A_{\frac{n}{n+2}} L^\alpha x \|.
\]
This completes the proof. \( \square \)

For the next theorem, in addition to (3.13), we assume that
\[
\| A_{\frac{n}{n+2}} L^\alpha y \| \geq c \delta \tag{3.21}
\]
for some \( c > 0 \). This assumption will be satisfied, if for example,
\[
\delta \leq \frac{\tilde{f}(s)}{c + \tilde{f}(s)} \| y \|, \quad \tilde{f}(s) := f \left( \frac{-s}{2s + 2a} \right),
\]
since, by \((3.13)\), we have \(\|\tilde{y}\| \geq \|\hat{y}\| - \delta\), and by the relation \((3.15)\),
\[
\|A_{\frac{2s+2a}{s+2a}} L^{-\frac{1}{s}} \tilde{y}\| \geq f \left( \frac{-s}{2s + 2a} \right) \|\tilde{y}\|,
\]
where \(f\) is as in \((2.9)\).

Now, the following theorem is a consequence of Theorem 3.1.

**THEOREM 3.2.** Assume that \((3.13)\) and \((3.21)\) are satisfied. Then there exist a unique \(\alpha := \alpha(\delta)\) satisfying
\[
\Phi(\alpha, \tilde{y}) = c\delta.
\]  

(3.22)

4. **Error Estimates**

To obtain Hölder type error bounds, that is, error bounds of the form
\[
\|\hat{x}_\alpha - \hat{x}\| = O(\delta^\tau)
\]
for some \(\tau\), we assume that the solution \(\hat{x}\) of \((2.2)\) satisfies the source condition (as in [1], [10])
\[
\hat{x} \in M_{\rho,t} := \{x \in H_t : \|x\|_t \leq \rho\}.
\]  

(4.23)

for some \(t > 0\).

**LEMMA 4.1.** Suppose that \(\hat{x}\) belongs to \(M_{\rho,t}\) for some \(t \leq s\), and \(\alpha := \alpha(\delta) > 0\) is the unique solution of \((3.22)\), where \(c > g\left(\frac{-s}{2s + 2a}\right)\). Then
\[
\alpha \geq c_0 \delta^{\frac{1 + \alpha}{1 + \alpha}}, \quad c_0 = \frac{c - g\left(\frac{-s}{2s + 2a}\right)}{g\left(\frac{s - 2t}{2s + 2a}\right) \rho}\]

(4.24)

**Proof.** Note that by \((3.20)\), Proposition 2.1 and \((2.11)\), we have
\[
\Phi(\alpha, \tilde{y}) \leq \alpha \|R_\alpha A_{\frac{2s}{s+2a}} L^{-\frac{1}{s}} \tilde{y}\| \\
\leq \alpha \|R_\alpha A_{\frac{2s}{s+2a}} L^{-\frac{1}{s}} (\tilde{y} - y)\| + \alpha \|R_\alpha A_{\frac{2s}{s+2a}} A_{\frac{s-2t}{2s+2a}} L^{-\frac{s-2t}{s+2a}} \hat{x}\| \\
\leq \alpha \|R_\alpha A_{\frac{2s}{s+2a}} L^{-\frac{1}{s}} (\tilde{y} - y)\| + \alpha \|R_\alpha A_{\frac{2s}{s+2a}} A_{\frac{s-2t}{2s+2a}} L^{-\frac{s-2t}{s+2a}} \hat{x}\| \\
\leq \|\alpha R_\alpha\| \|A_{\frac{2s}{s+2a}} L^{-\frac{1}{s}} (\tilde{y} - y)\| + \|\alpha R_\alpha A_{\frac{2s}{s+2a}}\| \|\tilde{y}\|\|A_{\frac{s-2t}{s+2a}} L^{-\frac{s-2t}{s+2a}} \hat{x}\| \\
\leq g \left( \frac{-s}{2s + 2a} \right) \delta + g \left( \frac{s - 2t}{2s + 2a} \right) \rho \alpha^{\frac{1 + \alpha}{1 + \alpha}}.
\]

(4.25)

Thus
\[
\left[ c - g\left(\frac{-s}{2s + 2a}\right) \right] \delta \leq g\left(\frac{s - 2t}{2s + 2a}\right) \rho \alpha^{\frac{1 + \alpha}{1 + \alpha}}.
\]
which implies
\[ \alpha \geq c_0 \delta^{\frac{s}{2s+2a}}, \quad c_0 = \frac{c - g \left( \frac{s}{2s+2a} \right)}{g \left( \frac{s}{2s+2a} \right) \rho}. \]
This completes the proof. \( \square \)

**THEOREM 4.2.** Under the assumptions in Lemma 4.1,
\[ \| \hat{x} - x_\alpha \| = O(\delta^\alpha), \quad \kappa := \frac{t}{t + a}. \]

*Proof.* Since \( x_\alpha \) is the solution of (2.6), we have
\[
\hat{x} - x_\alpha = \hat{x} - (A + \alpha L^*)^{-1} y \\
= \alpha L^{-\frac{\delta}{2}} (A_\alpha + \alpha I)^{-1} L^{\frac{\delta}{2}} \hat{x} \\
= \alpha L^{-\frac{\delta}{2}} R_\alpha L^{\frac{\delta}{2}} \hat{x}.
\]
Therefore by (3.15), we have
\[
f \left( \frac{s}{2s+2a} \right) \| \hat{x} - x_\alpha \| \leq \| \alpha A_\alpha^{\frac{-2t}{2s+2a}} R_\alpha L^{\frac{\delta}{2}} \hat{x} \| (4.26)
\]
To obtain an estimate for \( \| \alpha A_\alpha^{\frac{-2t}{2s+2a}} R_\alpha L^{\frac{\delta}{2}} \hat{x} \| \), first we shall make use the following moment inequality
\[
\| B^\alpha x \| \leq \| B^\alpha \| \| x \|^{1 - \frac{\delta}{2}}, \quad 0 \leq u \leq v \tag{4.27}
\]
where \( B \) is a positive selfadjoint operator. Precisely we use (4.27) with
\[
u = \frac{t}{a}, \quad v = 1 + \frac{t}{a}, \quad B = \alpha R_\alpha A_\alpha^{\frac{a}{2s+2a}}, \quad x = \alpha^{1 - \frac{t}{a}} R_\alpha^{1 - \frac{t}{a}} A_\alpha^{\frac{s-t}{2s+2a}} L^{\frac{\delta}{2}} \hat{x}.
\]
Then since
\[
\| x \| \leq \| A_\alpha^{\frac{s-t}{2s+2a}} L^{\frac{\delta}{2}} \hat{x} \| \leq g \left( \frac{s-2t}{2s+2a} \right) \| L^{\frac{\delta}{2}} \hat{x} \| \leq g \left( \frac{s-2t}{2s+2a} \right) \rho (4.28)
\]
we have
\[
\| \alpha A_\alpha^{\frac{s-t}{2s+2a}} R_\alpha L^{\frac{\delta}{2}} \hat{x} \| = \| B^\frac{\delta}{2} x \| \\
\leq \| B^{1 + \frac{t}{a}} \| \| x \|^{\frac{a}{2s+2a}} \tag{4.29}
\]
Further by (2.11) and (3.19);
\[
\| \alpha^2 R_\alpha^2 A_\alpha^{\frac{t}{2s+2a}} L^{-\frac{\delta}{2}} \hat{y} \| \leq \| \alpha^2 R_\alpha^2 A_\alpha^{\frac{s-t}{2s+2a}} L^{-\frac{\delta}{2}} (y - \hat{y}) \| + \| \alpha^2 R_\alpha^2 A_\alpha^{\frac{t}{2s+2a}} L^{-\frac{\delta}{2}} \hat{y} \|
\leq \delta + \Phi(\alpha, \hat{y}).
\]
Therefore if $\alpha := \alpha(\delta)$ is the unique solution of (3.22), then we have

$$\|\alpha^2 R_\alpha^2 A^\frac{s}{\pi + s} L^{-\frac{s}{\pi + s}} y\| \leq (1 + c)\delta.$$  (4.30)

Now the result follows from (4.26), (4.28), (4.29) and (4.30). \qed

**THEOREM 4.3.** Under the assumptions in Lemma 4.1,

$$\| \hat{x} - \hat{x}_\alpha \| = O(\delta^\kappa), \quad \kappa := \frac{t}{t + a}.$$ 

**Proof.** Let $x_\alpha$ and $\hat{x}_\alpha$ be the solutions of (2.6) and (2.7) respectively. Then by triangle inequality, (2.11) and Proposition 2.1,

$$\| \hat{x} - \hat{x}_\alpha \| \leq \| \hat{x} - x_\alpha \| + \| x_\alpha - \hat{x}_\alpha \|$$

$$= \| \hat{x} - x_\alpha \| + \| L^{-\frac{s}{\pi + s}} R_\alpha L^{-\frac{s}{\pi + s}} (y - \hat{y}) \|$$

$$\leq \| \hat{x} - x_\alpha \| + \frac{1}{f\left(\frac{s}{2s + 2a}\right)} \| A_s^\frac{s}{\pi + s} R_\alpha L^{-\frac{s}{\pi + s}} (y - \hat{y}) \|$$

$$\leq \| \hat{x} - x_\alpha \| + \frac{1}{f\left(\frac{s}{2s + 2a}\right)} \| A_s^\frac{s}{\pi + s} R_\alpha A_s^\frac{s}{\pi + s} L^{-\frac{s}{\pi + s}} (y - \hat{y}) \|$$

$$\leq \| \hat{x} - x_\alpha \| + \frac{1}{f\left(\frac{s}{2s + 2a}\right)} \delta \alpha^{-\frac{s}{\pi + s}}$$  (4.31)

The proof now follows from Lemma 4.1 and Theorem 4.2. \qed

**REMARK.** We observe that unlike the discrepancy principle in George and Nair [1], the discrepancy principle (3.14) gives the optimal order $O(\delta^{\frac{1}{t}})$ for all $0 < t \leq s$.

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