Spectral Mapping Theorem for Self Adjoint Operators

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**THEOREM 1.** Let $A \in B(X)$ be a self-adjoint operator on a Hilbert space $X$ and $p(t)$ be a polynomial with coefficients in $\mathbb{F}$. Then,

$$\sigma(p(A)) = \{ p(\lambda) : \lambda \in \sigma(A) \}.$$ 

**Proof.** By Theorem 10.14 in [1],

$$\{ p(\lambda) : \lambda \in \sigma(A) \} \subseteq \sigma(p(A)),$$

and for $\mathbb{F} = \mathbb{C}$, we have the equality

$$\{ p(\lambda) : \lambda \in \sigma(A) \} = \sigma(p(A)).$$

Hence, we need to prove

$$\sigma(p(A)) \subseteq \{ p(\lambda) : \lambda \in \sigma(A) \}$$

for the case $\mathbb{F} = \mathbb{R}$. So, let $p(t)$ be a polynomial with real coefficients. The result is obvious if $p(t)$ is a constant polynomial. Hence, assume that $p(t)$ is not constant. Let $\mu \in \sigma(p(A))$. We observe that, since $A$ is self adjoint and since coefficients of $p(t)$ are real numbers, $p(A)$ is also a self adjoint operator. Hence $\mu \in \mathbb{R}$. We consider two cases, namely,

**Case(i):** $p(t) - \mu$ does not have any non-real complex zeros, and

**Case(ii):** $p(t) - \mu$ has at least one non-real complex zero.

**Case(i):** In this case, there are (not necessarily distinct) real numbers $c,t_1,\ldots,t_k$ such that

$$p(t) - \mu = c \prod_{j=1}^{k} (t - t_j).$$

Then

$$p(A) - \mu I = c \prod_{j=1}^{k} (A - t_jI).$$

Since $p(A) - \mu I$ is not invertible, there exists $\ell \in \{1,\ldots,k\}$ such that $A - t_\ell I$ is not invertible. Thus, $\lambda := t_\ell$ is a spectral value of $A$, and $p(\lambda) = \mu$.

**Case(ii):** Suppose $\lambda$ is a non-real complex zero of $p(t) - \mu$. Since coefficients of $p(t) - \mu$ are real numbers, $\lambda$ is also a zero of $p(t) - \mu$. Hence, $(t - \lambda)(t - \bar{\lambda})$ is a factor of $p(t) - \mu$. Thus, $p(t) - \mu$ has

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1A slight modified form of the proof given in [1], Theorem 12.12, Page 386.
the representation

\[ p(t) - \mu = q(t) \prod_{j=1}^{m} (t - \lambda_j I)(t - \bar{\lambda}_j I), \]

where \( \lambda_1, \ldots, \lambda_m \) are non-real (not necessarily distinct) complex numbers and \( q(t) \) is a polynomial with real coefficients having no non-real zeros. Writing \( \lambda_j = \alpha_j + i\beta_j \) with \( \alpha_j, \beta_j \in \mathbb{R} \) and \( \beta_j \neq 0 \) and observing that

\[ (t - \lambda_j)(t - \bar{\lambda}_j) = [(t - \alpha_j) - i\beta_j][(t - \alpha_j) + i\beta_j] = (t - \alpha_j)^2 + \beta_j^2, \]

we have

\[ p(t) - \mu = q(t) \prod_{j=1}^{m} [(t - \alpha_j)^2 + \beta_j^2]. \]

Hence,

\[ p(A) - \mu I = q(A) \prod_{j=1}^{m} [(A - \alpha_j I)^2 + \beta_j^2 I]. \]

Since each \( (A - \alpha_j I)^2 \) is a positive operator, \( (A - \alpha_j I)^2 + \beta_j^2 I \) is invertible for every \( j \in \{1, \ldots, m\} \). Hence, \( q(A) \) is not invertible. Therefore, \( q(t) \) is a non-constant polynomial and \( q(t) \) is of the form

\[ q(t) = c \prod_{j=1}^{k} (t - t_j), \]

for some real numbers \( c, t_1, \ldots, t_k \). Consequently, we have

\[ p(A) - \mu I = c \prod_{j=1}^{k} (A - t_j I) \prod_{j=1}^{m} [(A - \alpha_j I)^2 + \beta_j^2 I]. \]

From this it follows, as in case (i), that there exists \( \ell \in \{1, \ldots, k\} \) such that \( A - t_\ell I \) is not invertible. Thus, \( \lambda := t_\ell \) is a spectral value of \( A \), and \( p(\lambda) = \mu \).

\[ \square \]

References