# A Spectral Characterization of Closed Range Operators

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## 1 Closed Range Operators

Operator equations of the form

\[ Tx = y, \]

where \( T : X \rightarrow Y \) is a linear operator between normed linear spaces, occur very naturally in applications. In order to assess the stability of the solutions, it is important to know the closedness of the range of \( T \) (cf. Nair [1], Chapter 14)

In this talk, we shall give discuss a characterization of the closedness of the range of \( T \) in terms of the spectrum of the operator \( T^*T \) when \( X \) and \( Y \) are Hilbert spaces. First we give some typical examples of operators of non-closed ranges.

## 2 Examples

**Example 1.** Consider the operator \( T : \ell^2 \rightarrow \ell^2 \) defined by

\[ T(\alpha_1, \alpha_2, \alpha_3, \ldots) = \left( \alpha_1, \frac{\alpha_2}{2}, \frac{\alpha_3}{3}, \ldots \right), \quad (\alpha_1, \alpha_2, \alpha_3, \ldots) \in \ell^2. \]

Clearly, each \( e_j \) belongs to \( R(T) \) so that \( R(T) \) is infinite dimensional. Is \( R(T) \) closed? If we take \( x_n \) defined by \( x_n(j) = \begin{cases} 1, & j \leq n, \\ 0, & j > n \end{cases} \), then we see that \( (Tx_n)(j) = \begin{cases} 1/j, & j \leq n, \\ 0, & j > n \end{cases} \) and \( (Tx_n) \) converges to \( y = (1, 1/2, 1/3, \ldots) \) which does not belong to \( R(T) \). Hence, \( R(T) \) is not closed in \( \ell^2 \). In this example, we note that each \( \lambda_j := 1/j \) is an eigenvalue of \( T \) and that \( \lambda_n \rightarrow 0 \) as \( n \rightarrow \infty \). Hence, 0 is an accumulation point of the eigen spectrum of \( T \).

\(^1\)A Talk at the National Seminar on Analysis, Topology and Applications at Chittor Govt. College, Palakkad, Kerala, 8-9 January 2005
**Question:** If 0 is an accumulation point of the eigen spectrum of \( T : \ell^2 \to \ell^2 \), then is \( R(T) \) not closed?

Before discussing the above question, let us consider a general example.

**Example 2.** Let \( X \) be a separable Hilbert space with orthonormal basis \( \{ u_n : n \in \mathbb{N} \} \). Let \( (\lambda_n) \) be a sequence of non-zero scalars such that \( \lambda_n \to 0 \) as \( n \to \infty \). Let \( T : X \to X \) be defined by

\[
T x = \sum_{j=1}^{\infty} \lambda_j \langle x, u_j \rangle u_j, \quad x \in X.
\]

Note that \( T \) is a bounded linear operator, each \( \lambda_j \) is an eigenvalue of \( T \) and 0 is an accumulation point of the eigen spectrum of \( T \). We also observe that \( T \) is one-one, and

\[
R(T) = \{ y \in X : \sum_{j=1}^{\infty} \frac{|\langle y, u_j \rangle|^2}{|\lambda_j|^2} < \infty \}.
\]

The last conclusion follows since for \( y \in X \), \( y = Tx \) for some \( x \in X \) if and only if

\[
\sum_{j=1}^{\infty} \langle y, u_j \rangle u_j = \sum_{j=1}^{\infty} \lambda_j \langle x, u_j \rangle u_j \iff \langle x, u_j \rangle = \frac{\langle y, u_j \rangle}{\lambda_j} \quad \forall j \in \mathbb{N}.
\]

Now, for \( y \in X \), if we take \( x_n := \sum_{j=1}^{n} \frac{\langle y, u_j \rangle}{\lambda_j} u_j \), then we have \( Tx_n = \sum_{j=1}^{n} \langle y, u_j \rangle u_j \) so that \( y = \lim_{n \to \infty} Tx_n \). Thus, \( R(T) \) is dense in \( X \).

Suppose the sequence \( (\lambda_n) \) is such that \( \sum_{n=1}^{\infty} |\lambda_n|^2 \) converges. Then it follows that \( y = \sum_{n=1}^{\infty} \lambda_n u_n \in \ell^2 \), but \( y \not\in R(T) \). Hence, \( R(T) \) is a proper dense subspace of \( X \).

In general also, \( R(T) \) is a proper dense subspace of \( X \), because, if \( R(T) = X \), then by bounded inverse theorem (cf. Nair [1], Chapter 5) \( T^{-1} : X \to X \) has to be continuous, which is not true. Indeed, image of the bounded sequence \( (u_n) \) under \( T^{-1} \) is \( (u_n/\lambda_n) \) which is not bounded. Thus, \( R(T) \) is a proper dense subspace of \( X \). Consequently, \( R(T) \) is not closed in \( X \).

Now, a slightly more general example than the above.

**Example 3.** Let \( X \) and \( Y \) be a separable Hilbert spaces with orthonormal bases \( \{ u_n : n \in \mathbb{N} \} \) and \( \{ v_n : n \in \mathbb{N} \} \) respectively. Let \( (\mu_n) \) be a sequence of non-zero scalars.
which converges to zero. Let $T : X \to Y$ be defined by

$$Tx = \sum_{j=1}^{\infty} \mu_j \langle x, u_j \rangle v_j, \quad x \in X.$$  

We observe that $T$ is a bounded linear operator and $Tu_j = \mu_j v_j$ for every $j \in \mathbb{N}$. Its adjoint $T^*$ is given by

$$T^* y = \sum_{j=1}^{\infty} \overline{\mu}_j \langle y, v_j \rangle u_j, \quad y \in Y.$$  

Hence,

$$T^* Tx = \sum_{j=1}^{\infty} |\mu_j|^2 \langle x, u_j \rangle u_j, \quad x \in X.$$  

Note that each $|\mu_j|^2$ is an eigenvalue of $T^*T$ which converges to 0. Taking $w_j := Tu_j / |\mu_j|$, it is seen that

$$Tu_j = |\mu_j| w_j \text{ and } T^* w_j = |\mu_j| u_j \quad \forall j \in \mathbb{N}.$$  

The scalars $|\mu_j|$ are called the singular values of $T$, and the vectors $u_j, w_j$ are called the associated singular vectors.

In this case also, we see that $T$ is one-one,

$$R(T) = \{ y \in X : \sum_{j=1}^{\infty} \frac{\langle y, v_j \rangle^2}{|\mu_j|^2} < \infty \},$$  

and $R(T)$ is dense in $Y$. The last part is seen by observing that

$$y = \lim_{n \to \infty} \sum_{j=1}^{n} \langle y, v_j \rangle v_j = \lim_{n \to \infty} T x_n \quad \forall y \in Y,$$

where $x_n := \sum_{j=1}^{n} \frac{\langle y, v_j \rangle}{\mu_j} u_j$. Again, by making use of bounded inverse theorem, we can conclude that $R(T) \neq Y$, and consequently, $R(T)$ is not closed in $Y$.

### 3 The Spectral Characterization

In all the three examples in last section, we observed that 0 is an accumulation point of the eigen spectrum of $T^*T$, and $R(T)$ is not closed. This observation motivates the following conjecture:
Suppose $X$ and $Y$ are Hilbert spaces and $T : X \to Y$ is a bounded linear operator. Then $R(T)$ is closed if and only if $0$ is an accumulation point of the eigen spectrum of $T^*T$.

But, there are bounded operators $T$ having no singular values. For example, The operator $T : L^2[0, 1] \to L^2[0, 1]$ defined by
\[
(Tx)(t) = tx(t), \quad x \in L^2[0, 1], \text{ a.a. } t \in [0, 1],
\]
has no singular values. In fact, this operator is self adjoint, i.e., $T^* = T$, and the eigen spectrum of $T$ and $T^*T = T^2$ are empty set. Hence, the conjecture can be void for a general bounded operator.

However, for every compact operator $T$, the eigen spectrum of $T^*T$ is non-empty, and it is a countable set. In this case, it is a known fact that $0$ is an accumulation point of $T^*T$ if and only if $R(T^*T)$ is of infinite rank and hence $R(T^*T)$ is not closed. For all these results, one may refer Nair ([1], Chapters 9, 10, 12).

What about for a general bounded operator? Well, we should not restrict to eigen spectrum alone. So, we would like to know if the following result is true.

**THEOREM 3.1** Suppose $X$ and $Y$ are Hilbert spaces and $T : X \to Y$ is a bounded linear operator. Then $R(T)$ is closed if and only if $0$ is not an accumulation point of the spectrum of $T^*T$.

This result is true. We have given three different proofs for this result in [2]. In this talk we give another proof (part of which is included in [2]) for the same which has direct bearing on ill-posed operator equations. First let us recall some definitions and results from operator theory.

Let $X$ and $Y$ be Hilbert spaces and $T : X \to Y$ be a bounded linear operator. Recall that the adjoint $T^*$ of $T$ is the unique bounded linear operator from $Y$ to $X$ such that
\[
\langle Tx, y \rangle = \langle x, T^*y \rangle \quad \forall x \in X, y \in Y,
\]
whose existence is guaranteed by Riesz representation theorem (cf. [1], Chapter 3).
For a bounded operator $A : X \to X$, the spectrum $\sigma(A)$ is the set of all scalars $\lambda$ such that $A + \lambda I$ is not bijective. It is known that for a self adjoint operator $A$ (i.e., if $A^* = A$),
\[
r_{\sigma}(A) := \sup\{|\lambda| : \lambda \in \sigma(A)\} = \|A\|,
\]
and if $A$ is a self a self adjoint and positive operator (i.e., $\langle Ax, x \rangle \geq 0$ for every $x \in X$), then $\sigma(A) \subseteq [0, \infty)$. Moreover, $\sigma(A + \alpha I) = \{\lambda + \alpha : \lambda \in \sigma(A)\}$.

A few more spectral results will be used during the course of proofs of the results.

For $\alpha > 0$, let
\[
R_\alpha := (T^*T + \alpha I)^{-1}T^*.
\]

A crucial result for the proof of Theorem 3.1 is the following.

**Proposition 3.2** The family $\{R_\alpha : \alpha > 0\}$ is uniformly bounded if and only if $R(T)$ is closed.

Let us first make use of the above proposition and prove Theorem 3.1.

**Proof of Theorem 3.1:** We may observe that
\[
R_\alpha^* R_\alpha = (T^*T + \alpha I)^{-2}T^*T \quad \forall \alpha > 0.
\]
Since $R_\alpha^* R_\alpha$ is self adjoint, it follows that
\[
\|R_\alpha\|^2 = \|R_\alpha^* R_\alpha\| = r_{\sigma}(R_\alpha^* R_\alpha) = \sup\left\{\frac{\lambda}{(\lambda + \alpha)^2} : \lambda \in \sigma(T^*T)\right\}.
\]
Taking $0 < \alpha < \|T\|^2$ and
\[
f_\alpha(t) := \frac{t}{(t + \alpha)^2}, \quad t \in (0, \|T\|^2],
\]
we see that $f_\alpha$ attains its maximum at $t = \alpha$, i.e.,
\[
\sup_{0< t \leq \|T\|^2} f_\alpha(t) = f_\alpha(\alpha) = \frac{1}{4\alpha}.
\]

Now suppose that $0$ is an accumulation point of $\sigma(T^*T)$. Then there exists a sequence $(\lambda_n)$ of distinct elements in $\sigma(T^*T)$ such that $\lambda_n \to 0$ as $n \to \infty$. Hence,
\[
\|R_{\lambda_n}\|^2 = f_{\lambda_n}(\lambda_n) = \frac{1}{4\lambda_n} \to \infty \text{ as } n \to \infty.
\]
Thus, the family \( \{ R_\alpha : \alpha > 0 \} \) is not uniformly bounded. Hence, by Proposition 3.2, \( R(T) \) is not closed.

Conversely, suppose 0 is not an accumulation point of \( \sigma(T^*T) \). Then there exists \( \gamma > 0 \) such that

\[ \sigma(T^*T) \subseteq \{ 0 \} \cup [\gamma, \|T\|^2]. \]

Note that \( f_\alpha(0) = 0 \), and

\[ f_\alpha(\lambda) \leq \frac{1}{\lambda} \leq \frac{1}{\gamma} \]

whenever \( 0 \neq \lambda \in \sigma(T^*T) \).

Consequently,

\[ \|R_\alpha\|^2 = \sup_{0 < \lambda \leq \|T\|^2} f_\alpha(\lambda) \leq \frac{1}{\gamma}. \]

Thus, the family \( \{ R_\alpha : \alpha > 0 \} \) is uniformly bounded. Again, by Proposition 3.2, \( R(T) \) is closed. \( \blacksquare \)

In order to prove Proposition 3.2, we shall make use of the concept of the generalized inverse of a bounded linear operator.

**Lemma 3.3** For every \( y \in R(T) + R(T)^\perp \), there exists a unique \( x_y \in N(T)^\perp \) such that \( T^*Tx_y = T^*y \). The map \( T^\dagger : R(T) + R(T)^\perp \rightarrow X \) defined by \( T^\dagger y = x_y \) for \( y \in R(T) + R(T)^\perp \), is a closed densely defined operator, and it is a bounded operator if and only if \( R(T) \) is closed in \( Y \).

The first part of the proof of the above lemma follows by making use of the projection theorem (cf. [1], Chapter 2) judiciously, and hence left as an exercise. The second part of the lemma is a consequence of closed graph theorem.

The operator \( T^\dagger \) defined in Lemma 3.3 is called the (Moore-Penrose) **generalized inverse** of \( T \). It can be seen (cf. [1], Chapter 14) that

\[ T^\dagger y = (T^*T|_{N(T)^\perp})^{-1}T^*y \quad \forall y \in R(T) + R(T)^\perp. \]

We may observe that if \( T \) is invertible, then \( T^\dagger = T^{-1} \).

**Lemma 3.4** For every \( y \in R(T) + R(T)^\perp \),

\[ R_\alpha y \rightarrow T^\dagger y \quad \text{as} \quad \alpha \rightarrow 0. \]
Proof. We observe that for \( y \in R(T) + R(T)\perp, \hat{x} = T^\dagger y \) satisfies \( T^\ast \hat{x} = T^\ast y \) so that
\[
(T^\ast T + \alpha I)\hat{x} = T^\ast y + \alpha \hat{x}
\]
so that
\[
\hat{x} = (T^\ast T + \alpha I)^{-1}T^\ast y + \alpha (T^\ast T + \alpha I)^{-1} \hat{x} = R_\alpha y + \alpha (T^\ast T + \alpha I)^{-1} \hat{x}.
\]
Thus,
\[
\hat{x} - R_\alpha y = \alpha (T^\ast T + \alpha I)^{-1} \hat{x}.
\]
Now, let \( A_\alpha := \alpha (T^\ast T + \alpha I)^{-1} \). Then we have \( \|A_\alpha\| \leq 1 \) for all \( \alpha > 0 \). We may also observe that \( R(T^\ast T) \) is a dense subspace of \( N(T)^\perp \) and \( \hat{x} = T^\dagger y \in N(T)^\perp \) for every
\( y \in R(T) + R(T)\perp \). Now, for \( v \in R(T^\ast T) \), if \( v = T^\ast Tu \), then
\[
A_\alpha v = \alpha (T^\ast T + \alpha I)^{-1}T^\ast Tu \to 0 \quad \text{as} \quad \alpha \to 0.
\]
Hence, it follows (cf. [1], Theorem 3.11) that \( A_\alpha y \to 0 \) as \( \alpha \to 0 \) for every \( v \in N(T)^\perp \).

In particular,
\[
\hat{x} - R_\alpha y = \alpha (T^\ast T + \alpha I)^{-1} \hat{x} = A_\alpha \hat{x} \to 0 \quad \text{as} \quad \alpha \to 0.
\]
This completes the proof. ■

Proof of Proposition 3.2: Suppose \( \{R_\alpha : \alpha > 0\} \) is uniformly bounded. Since \( R(T) + R(T)^\perp \) is dense in \( Y \), by Lemma 3.4 and ([1], Theorem 3.11), \( \lim_{\alpha \to 0} R_\alpha y \) exists for every \( y \in Y \), and the limiting operator is a bounded operator. Hence, its restriction to \( R(T) + R(T)^\perp \), which is \( T^\dagger \) also must be a bounded operator. This implies, by Lemma 3.3 and ([1], Theorem 3.11), that \( R(T) \) is closed.

Conversely, suppose that \( R(T) \) is closed in \( Y \). Then \( R(T) + R(T)^\perp = Y \). Hence, Lemma 3.4 together with Uniform Boundedness Theorem (cf. [1], Chapter 6) implies that \( \{R_\alpha : \alpha > 0\} \) is uniformly bounded. ■

References