AN ELEMENTARY PROOF FOR A CHARACTERIZATION OF *-ISOMORPHISMS

S. H. KULKARNI, M.T. NAIR, AND M.N.N. NAMBOODIRI

ABSTRACT. We give an elementary proof for a result which characterizes onto *-isomorphisms of the algebra $BL(H)$ of all the bounded linear operators on a Hilbert space $H$. A known proof of this result (cf. Arveson, 1976) relies on the theory of irreducible representations of $C^*$-algebras, whereas the proof given by us is based on elementary properties of operators on a Hilbert space which can be found in any introductory text on Functional Analysis.

1. Introduction

The aim of this note is to give an elementary proof of the following theorem that characterizes onto *-isomorphisms of the algebra $BL(H)$ of all the bounded linear operators on a Hilbert space $H$.

Theorem 1.1. Let $H$ and $K$ be Hilbert spaces and $\Phi$ be a *-isomorphism from $BL(H)$ onto $BL(K)$. Then there exists a unitary operator $U : H \to K$ such that $\Phi(T) = UTU^*$ for all $T \in BL(H)$.

The classical proof of this theorem relies on the theory of irreducible representations of $C^*$-algebras (see, for example [1]). Our proof uses only very basic properties of the operators between Hilbert spaces, which can be found in any introductory text on Functional Analysis, for example [2]. Some of these properties are also mentioned explicitly at the beginning of the next section. In particular, we do not use any concept involving the spectrum of an operator.

2. Main result

Let $H, K$ be a Hilbert spaces. We denote by $BL(H, K)$ the set of all bounded linear maps from $H$ to $K$, and $BL(H, H)$ will be denoted by $BL(H)$. For $T \in BL(H, K)$, $T^*$ will denote the adjoint of $T$, $N(T)$, the null space of $T$ and $R(T)$, the range of $T$.

In what follows, we shall make repeated use of some well known facts about orthogonal projections. For a ready reference, we collect those facts in the following theorem. Proofs can be found in any introductory text on Functional Analysis (e.g., [2]).

1991 Mathematics Subject Classification. Primary 47L10; Secondary 47L30.

Key words and phrases. *-isomorphism, unitary operator, orthogonal projection.
Theorem 2.1. Let \( H, K \) be a Hilbert spaces.

1. Let \( P \in BL(H) \) be a projection, that is, \( P^2 = P \). Then \( P \) is orthogonal (that is \( R(P) \perp N(P) \)) \( \iff \) \( P^* = P \) (that is, \( P \) is self adjoint).

2. Let \( P, Q \) be orthogonal projections in \( BL(H) \). Then,
   - (a) \( R(P) \subseteq R(Q) \iff PQ = P = QP \).
   - (b) \( R(P) \perp R(Q) \iff PQ = 0 = QP \).

In addition to the above, we shall need some properties of operators and projections of rank one. We prove these in the following proposition.

Proposition 2.2. Let \( H \) be a Hilbert space. For \( x, y \in H \) define the operator \( T_{x,y} \) on \( H \) by

\[
T_{x,y}(u) = \langle u, y \rangle x, \quad u \in H,
\]

and let \( P_x := T_{x,x} \). Then the following statements hold.

1. For \( x, y \in H, T_{x,y} \in BL(H) \) and \( \text{Rank}(T_{x,y}) = 1 \).
2. For \( x, y \in H \), and scalars \( \alpha, \beta \), \( T_{\alpha x, \beta y} = \alpha \beta T_{x,y} \).
   In particular, \( P_{\lambda x} = P_x \) for all scalars \( \lambda \) with \( |\lambda| = 1 \)
3. If \( x \in H \) with \( \|x\| = 1 \), then \( P_x \) is an orthogonal projection of rank 1.
4. If \( P \in BL(H) \) is an orthogonal projection of rank 1, then there exists \( x \in H \) with \( \|x\| = 1 \) such that \( P = P_x \).
5. For \( x, y \in H \) with \( \|x\| = 1 = \|y\| \),
   \[
   \langle x, y \rangle = 0 \iff P_x P_y = 0 = P_y P_x.
   \]
6. For \( x, y, z \in H, T_{x,z} T_{y,x} = \langle y, z \rangle P_x \).
7. For \( x, y \in H \), \( (T_{x,y})^* = T_{y,x} \).
8. Let \( x, y, z \in H \) with \( \|x\| = 1 = \|y\| \). Then
   \[
   T_{x,y} P_z = \begin{cases} 
   0, & \text{if } \langle z, y \rangle = 0, \\
   T_{x,y}, & \text{if } z = y.
   \end{cases}
   \]

Also
\[
P_x T_{x,y} = T_{x,y}, \quad T_{x,y} T_{y,z} = T_{x,z}.
\]

In particular, \( T_{x,y} T_{y,x} = P_x \).
9. Let \( T \in BL(H) \) be of rank 1. Then there exist \( x, y \in H \) with \( \|x\| = 1 \) such that \( T = T_{x,y} \). Further, \( x \) can be chosen as any element of norm 1 in \( R(T) \) and in that case \( y = T^*(x) \).

Proof. Results in (1) and (2) are routine verification.

3. Let \( x \in H \) with \( \|x\| = 1 \). Then \( P_x(x) = x \). Hence for all \( u \in H \),
   \[
P_x^2(u) = P_x(\langle u, x \rangle x) = \langle u, x \rangle x = P_x(u).
   \]

Thus \( P_x \) is a projection. Also \( R(P_x) = \text{span}(\{x\}) \). Hence, \( \text{Rank}(P_x) = 1 \). Next, for \( u \in H, u \in N(P_x) \iff \langle u, x \rangle x = 0 \). This shows that \( N(P_x) \perp R(P_x) \).
We observe that $x (9)$. Let $x, y, u, v (7)$. For $x, y, z, u (6)$. For $T (5)$. For $x, y (8)$. For $\Phi (3, 1) = \spann \{ x \}$. Let $u \in H$. Then

$$P(u) = \langle P(u), x \rangle x = \langle u, P(x) \rangle x = \langle u, x \rangle x = P_x (u).$$

Thus $P = P_x$.

(5). For $x, y \in H$ with $\| x \| = 1 = \| y \|$,

$$\langle x, y \rangle = 0 \iff R(P_x) \perp R(P_y) \iff P_x P_y = 0 = P_y P_x.$$

(6). For $x, y, z, u \in H$,

$$T_{x,z} T_{y,z}(u) = T_{x,z}(\langle u, x \rangle y) = \langle u, x \rangle T_{x,z}(y) = \langle u, x \rangle \langle y, z \rangle x = \langle y, z \rangle P_x (u).$$

(7). For $x, y, u, v \in H$,

$$\langle u, (T_{x,y})^*(v) \rangle = \langle T_{x,y}(u), v \rangle = \langle u, y \rangle \langle x, v \rangle = \langle u, \overline{\langle x, v \rangle} y \rangle = \langle u, \langle v, x \rangle y \rangle = \langle u, T_{y,x}(v) \rangle$$

(8). For $x, y, z, u \in H$ with $\| x \| = 1 = \| y \|$, and $z, u \in H$,

$$T_{x,y} P_z (u) = T_{x,y}(\langle u, z \rangle z) = \langle u, z \rangle T_{x,y}(z) = \langle u, z \rangle \langle z, y \rangle x.$$

We observe that

$$\langle u, z \rangle \langle z, y \rangle x = \begin{cases} 0, & \text{if } \langle z, y \rangle = 0, \\ \langle u, y \rangle x, & \text{if } z = y. \end{cases}$$

Other assertions can be proved in a similar way.

(9). Let $x \in R(T)$ with $\| x \| = 1$ and $y = T^*(x)$. Since $\text{Rank}(T) = 1$, we have $R(T) = \spann \{ x \}$, and hence for any $u \in H$,

$$T(u) = \langle T(u), x \rangle x = \langle u, T^*(x) \rangle x = \langle u, y \rangle x = T_{x,y}(u).$$

Thus $T = T_{x,y}$.

We now proceed to prove the main theorem. Recall that for Hilbert spaces $H, K$, a linear map $\Phi : BL(H) \to BL(K)$ is called an *isomorphism* if it is one-one and $\Phi(TS) = \Phi(T) \Phi(S)$ for all $T, S \in BL(H)$. An isomorphism $\Phi$ is called a *-isomorphism if $\Phi(T^*) = (\Phi(T))^*$ for all $T \in BL(H)$.

**Theorem 2.3.** Let $H$ and $K$ be Hilbert spaces and $\Phi$ be a *-isomorphism from $BL(H)$ onto $BL(K)$. Then there exists a unitary operator $U : H \to K$ such that $\Phi(T) = UTU^*$ for all $T \in BL(H)$.

**Proof.** We divide the proof into four steps.

**Step 1.** For each $x \in H$ with $\| x \| = 1$, there exists $\tilde{x} \in K$ with $\| \tilde{x} \| = 1$ such that $\Phi(P_x) = P_{\tilde{x}}$:  


Let $x \in H$ with $\|x\| = 1$. By Proposition 2.2, $P_x$ is an orthogonal projection of rank 1. Then
\[
(\Phi(P_x))^2 = \Phi((P_x)^2) = \Phi(P_x).
\]
Hence $\Phi(P_x)$ is a projection. Also since
\[
(\Phi(P_x))^* = \Phi((P_x)^*) = \Phi(P_x),
\]
$\Phi(P_x)$ is a self-adjoint and hence orthogonal projection.

We claim that $\Phi(P_x)$ is of rank 1. If not, there exist $y, z \in R(\Phi(P_x))$ such that $\|y\| = 1 = \|z\|$ and $\langle y, z \rangle = 0$. Consider the orthogonal projections $P_y, P_z$ in $BL(K)$. Since
\[
R(P_y) = \text{span}\{\{y\}\} \subseteq R(\Phi(P_x)),
\]
we have $P_y \Phi(P_x) = P_y$. Since $\Phi^{-1}$ is also a *-isomorphism, we have $\Phi^{-1}(P_y)$ is an orthogonal projection and $\Phi^{-1}(P_y) P_x = \Phi^{-1}(P_y)$. Hence
\[
R(\Phi^{-1}(P_y)) \subseteq R(P_x).
\]
Similarly,
\[
R(\Phi^{-1}(P_z)) \subseteq R(P_x).
\]
Now there exist nonzero elements $u \in R(\Phi^{-1}(P_y))$ and $v \in R(\Phi^{-1}(P_z))$. Then $u, v \in R(P_x) = \text{span}\{\{x\}\}$. On the other hand, since $\langle y, z \rangle = 0$, $P_y P_z = 0$, so that $\Phi^{-1}(P_y) \Phi^{-1}(P_z) = 0$. Thus
\[
R(\Phi^{-1}(P_y)) \perp R(\Phi^{-1}(P_z)),
\]
consequently, $u \perp v$, a contradiction, proving the claim. Thus $\Phi(P_x)$ is an orthogonal projection of rank 1. Hence by Proposition 2.2, there exists $\tilde{x} \in K$ with $\|\tilde{x}\| = 1$ such that $\Phi(P_x) = P_x$.

**Step 2.** The construction of the unitary map $U$:

Fix $x_0 \in H$ with $\|x_0\| = 1$. Using Step 1, choose $\tilde{x}_0 \in K$ with $\|\tilde{x}_0\| = 1$ such that $\Phi(P_{x_0}) = P_{x_0}$. Now define $U : H \to K$ by
\[
U(y) := \Phi(T_{y,x_0})(\tilde{x}_0), \quad y \in H.
\]
Note that
\[
U(x_0) := \Phi(T_{x_0,x_0})(\tilde{x}_0) = \Phi(P_{x_0})(\tilde{x}_0) = P_{x_0}(\tilde{x}_0) = \tilde{x}_0.
\]
Clearly, $U$ is linear by Proposition 2.2. Now to prove that $U$ is unitary, consider $y, z \in H$. Then,
\[
\langle U(y), U(z) \rangle = \langle (\Phi(T_{y,x_0})(\tilde{x}_0), \Phi(T_{z,x_0})(\tilde{x}_0)) \rangle
\]
\[
= \langle (\Phi(T_{z,x_0}))^* \Phi(T_{y,x_0})(\tilde{x}_0), \tilde{x}_0 \rangle
\]
\[
= \langle (\Phi(T_{z,x_0}))^* \Phi(T_{y,x_0})(\tilde{x}_0), \tilde{x}_0 \rangle
\]
\[
= \langle \Phi(T_{z,x_0})^* \Phi(T_{y,x_0})(\tilde{x}_0), \tilde{x}_0 \rangle
\]
\[
= \langle \Phi((y, z)P_{x_0})(\tilde{x}_0), \tilde{x}_0 \rangle
\]
\[
= \langle (y, z) P_{x_0}(\tilde{x}_0), \tilde{x}_0 \rangle
\]
\[
= \langle (y, z)(\tilde{x}_0, \tilde{x}_0) = \langle y, z \rangle.
\]
AN ELEMENTARY PROOF FOR A CHARACTERIZATION OF *-ISOMORPHISMS

This proves that $U$ preserves inner products. Next we prove that $U$ is onto. This will imply that $U$ is unitary. For this, let $v \in K$ and consider

$$y := (\Phi^{-1}(T_{v,x_0}))(x_0)$$

Then, for $u \in H$,

$$T_{y,x_0}(u) = \langle u, x_0 \rangle y$$

$$= \langle u, x_0 \rangle \Phi^{-1}(T_{v,x_0})(x_0)$$

$$= \Phi^{-1}(T_{v,x_0})P_{x_0}(u)$$

Hence,

$$T_{y,x_0} = \Phi^{-1}(T_{v,x_0})P_{x_0}$$

Thus,

$$\Phi(T_{y,x_0}) = T_{v,x_0} \Phi(P_{x_0}) = T_{v,x_0}P_{x_0} = T_{v,x_0}$$

The last equality follows from Proposition 2.2 (8). Now

$$U(y) := \Phi(T_{y,x_0})(\tilde{x}_0) = T_{v,x_0}(\tilde{x}_0) = (\tilde{x}_0, \tilde{x}_0)v = v$$

This proves that $U$ is onto and hence unitary.

**Step 3.** $\Phi(T) = UTU^*$ for all $T \in BL(H)$ with $\text{Rank}(T) = 1$.

First we prove this for $T = T_{x_0,y}$ for $y \in H$. So, let $y \in H$. Then, for all $u \in H$,

$$\Phi(T_{x_0,y})(U(u)) = \Phi(T_{x_0,y})\Phi(T_{u,x_0})(\tilde{x}_0)$$

$$= \Phi(T_{x_0,y}T_{u,x_0})(\tilde{x}_0) = \Phi(\langle u, y \rangle P_{x_0})(\tilde{x}_0)$$

$$= \langle u, y \rangle \Phi(P_{x_0})(\tilde{x}_0)$$

$$= \langle u, y \rangle P_{x_0}(\tilde{x}_0)$$

$$= \langle u, y \rangle \tilde{x}_0$$

$$= \langle u, y \rangle U(x_0)$$

$$= U(\langle u, y \rangle x_0)$$

$$= UT_{x_0,y}(u).$$

Thus $\Phi(T_{x_0,y})U = UT_{x_0,y}$, that is $\Phi(T_{x_0,y}) = UT_{x_0,y}U^*$.

Next, let $T \in BL(H)$ with $\text{Rank}(T) = 1$. By Proposition 2.2 (9), there exist $x, y \in H$ with $\|x\| = 1$ such that $T = T_{x,y}$. But,

$$T_{x,y} = T_{x_0,x}T_{x_0,y} = (T_{x_0,x})^*T_{x_0,y}.$$ 

Hence

$$\Phi(T) = (\Phi(T_{x_0,x}))^*\Phi(T_{x_0,y})$$

$$= (UT_{x_0,x}U^*)^*UT_{x_0,y}U^*U(T_{x_0,x})^*U^*UT_{x_0,y}U^*$$

$$= UT_{x_0,x}T_{x_0,y}U^*$$

$$= UT_{x,y}U^*$$

$$= UTU^*.$$

**Step 4.** $\Phi(T) = UTU^*$ for all $T \in BL(H)$.  


Let $T \in BL(H)$ and let $x \in H$ with $\|x\| = 1$. Then $\text{Rank}(TP_x)$ is 0 or 1. Hence by Step 3,

$$UTP_xU^* = \Phi(TP_x) = \Phi(T)\Phi(P_x) = \Phi(T)UP_xU^*.$$ 

Now evaluating both sides of the above equation at $U(x)$ and observing that $U^*U(x) = x$ and $P_x(x) = x$, we get $UT(x) = \Phi(T)U(x)$. Since this holds for all $x \in H$ with $\|x\| = 1$, we have $UT = \Phi(T)U$, that is, $\Phi(T) = UTU^*$.

This completes the proof of the theorem.

**Remark 2.4.** A careful examination of the proof of Theorem 2.3 shows that the proof works even if we replace $BL(H)$ by a *-subalgebra $A \subseteq BL(H)$ and $BL(K)$ by a *-subalgebra $B \subseteq BL(K)$ such that $A$ and $B$ contain all operators of rank 1. In particular, we can take $A = CL(H)$, the algebra of all compact operators on $H$, and $B = CL(K)$, the algebra of all compact operators on $K$. In fact, in [1] this theorem is first proved for the *-isomorphism of $CL(H)$ and then extended to the case of the *-isomorphism of $BL(H)$.

**Remark 2.5.** Note that we have not used the continuity of $\Phi$ in our proof of Theorem 2.3. It is a consequence of Theorem 2.3. Usual proof of this fact involves spectral considerations, whereas our proof does not.

**References**


Department of Mathematics, Indian Institute of Technology - Madras, Chennai 600036

E-mail address: shk@iitm.ac.in

Department of Mathematics, Indian Institute of Technology - Madras, Chennai 600036

E-mail address: mtnair@iitm.ac.in

Department of Mathematics, Cochin University of Science and Technology, Kochi-682002

E-mail address: nambu@cusat.ac.in