Convolutions of Certain Analytic Functions

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Abstract. Ruscheweyh and Sheil-Small proved the Polya-Schoenberg conjecture that the classes of convex functions, starlike functions and close-to-convex functions are closed under convolution with convex functions. By making use of this result, the radii of starlikeness of order $\alpha$, parabolic starlikeness, and strong starlikeness of order $\gamma$ of the convolution between two starlike functions are determined. Similar convolution results for two classes of analytic functions are also obtained.

Keywords. Starlike, parabolic starlike, strongly starlike, radius problem, convolution.

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1. Introduction

Let $A$ be the class of all functions analytic in $D = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by $f(0) = 0 = f'(0) - 1$. Let $S$ be the subclass of $A$ consisting of univalent functions. For $0 \leq \alpha < 1$, let $S^*(\alpha)$ and $C(\alpha)$ be the subclasses of $S$ consisting of starlike functions of order $\alpha$ and convex functions of order $\alpha$, respectively. A starlike or convex function of order 0 is respectively called starlike or convex function, and is denoted by $S^*(0) = S^*$ and $C(0) = C$. The class $S^*_\gamma$ of strongly starlike functions of order $\gamma$, $0 < \gamma \leq 1$, consists of $f \in S$ satisfying the inequality

$$\left| \arg \left( \frac{zf''(z)}{f'(z)} \right) \right| < \frac{\gamma \pi}{2}, \quad z \in D.$$

The function $f \in S$ is uniformly convex if for every circular arc $\gamma$ contained in $D$ with center $\zeta \in D$ the image arc $f(\gamma)$ is convex. The class $UCV$ of all uniformly convex functions was introduced by Goodman [1]. Rønning [9], as well as Ma and Minda [5], independently proved that

$$f \in UCV \iff \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \left| \frac{zf''(z)}{f'(z)} \right|, \quad z \in D.$$
Rønning introduced a corresponding class of starlike functions called parabolic starlike functions. A function \( f \in \mathcal{A} \) is parabolic starlike if
\[
\Re \left( \frac{zf'(z)}{f(z)} \right) > \left| \frac{zf'(z)}{f(z)} - 1 \right|, \quad z \in \mathbb{D}.
\]
The class of all such functions is denoted by \( \mathcal{S}_P \).

Let \( \mathcal{S}_L \) be the class of functions \( f \in \mathcal{A} \) satisfying the inequality
\[
\left| \left( \frac{zf'(z)}{f(z)} \right)^2 - 1 \right| < 1, \quad z \in \mathbb{D}.
\]
Thus a function \( f \) is in the class \( \mathcal{S}_L \) if \( zf'(z)/f(z) \) lies in the region bounded by the right-half of the lemniscate of Bernoulli given by \( |w^2 - 1| < 1 \). This class \( \mathcal{S}_L \) was introduced by Sokół and Stankiewicz [15].

For two analytic functions \( f, g \in \mathcal{A} \), their convolution or Hadamard product, denoted by \( f \ast g \), is defined by
\[
(f \ast g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n.
\]
Pólya and Schoenberg [7] conjectured that the class of convex functions \( \mathcal{C} \) is preserved under convolution with convex functions: \( f, g \in \mathcal{C} \Rightarrow f \ast g \in \mathcal{C} \). In 1973, Ruscheweyh and Sheil-Small [11] (see also [12]) proved the Polya-Schoenberg conjecture. In fact, they also proved that the classes of starlike functions and close-to-convex functions are closed under convolution with convex functions. The proofs of these facts are evident from the following result which is also needed for our investigation.

**Theorem 1.2.** [12, Theorem 2.4] If \( f \in \mathcal{S}^* \) and \( \varphi \in \mathcal{C} \), then
\[
\frac{\varphi \ast f F}{\varphi \ast f} (\mathbb{D}) \subset \overline{\text{co}}(F(\mathbb{D})),
\]
for any function \( F \) analytic in \( \mathbb{D} \), where \( \overline{\text{co}}(F(\mathbb{D})) \) denotes the closed convex hull of \( F(\mathbb{D}) \).

The radius of a property \( \mathcal{P} \) of functions in a set \( \mathcal{M} \) is the largest number \( R \) such that every function in the set \( \mathcal{M} \) has the property \( \mathcal{P} \) in each disk \( \mathbb{D}_r = \{ z \in \mathbb{D} : |z| < r \} \) for every \( r < R \). The convolution of the Koebe function \( k(z) = z/(1-z)^2 \) with itself is not univalent. Thus, the convolution of two univalent (or starlike) functions need not be univalent. Since the radius of convexity of functions in the class \( \mathcal{S}^* \) is \( 2 - \sqrt{3} \), a result of Ruscheweyh and Sheil-Small showed that the radius of starlikeness of the convolution between two starlike functions is \( 2 - \sqrt{3} \) (see [4]). Silverman [14] has determined the radius of univalence for the convolution of a normalized univalent function with a close-to-convex function. He also found a lower bound for the radius of univalence of convolution between two univalent functions.

By making use of Theorem 1.2, the \( \mathcal{S}_P \)-radius (and the \( \mathcal{S}^*(\alpha), \mathcal{S}_L, \) and \( \mathcal{S}_L^* \) radii) is determined for the convolution of two starlike functions. Certain classes of analytic functions are also proved to be closed under convolution with convex functions.
2. Convolution of two starlike functions

Rønning [10] proved that the class $S_ρ$ is closed under convolution with a starlike function of order 1/2. However, the convolution of two starlike functions need not be in the class $S_ρ$. Therefore it is natural to determine the $S_ρ$, $S^{\alpha}_\gamma$, and $S_{\gamma}$ radii of the convolution $f * g$ between two starlike functions $f$ and $g$. We do this in Theorem 2.1.

**Theorem 2.1.** Let $f, g \in S^\ast$ and $h_\rho(z) := (f * g)(\rho z)/\rho$. Then

(a) $h_\rho \in S_ρ$ for $0 \leq \rho \leq (4 - \sqrt{13})/3 \approx 0.13148$,

(b) $h_\rho \in S^{\alpha}_\gamma$ for $0 \leq \rho \leq (2 - \sqrt{3 + \alpha^2})/(1 + \alpha)$,

(c) $h_\rho \in S_\gamma$ for $0 \leq \rho \leq (\sqrt{5} - 2)(\sqrt{7} - 1) \approx 0.09778$,

(d) $h_\rho \in S_{\gamma}$ for $0 \leq \rho \leq (2 - \sqrt{4 - b^2})/b$ where $b = \sin(\pi \gamma/2)$.

The upper bound for $\rho$ in each case is sharp.

**Proof.** Let $H(z) = z + \sum_{n=2}^{\infty} n^2 z^n$ and consider the disk containing the values $zH'(z)/H(z)$. The function $H$ in closed form is

$$H(z) = \frac{z(1+z)}{(1-z)^3}.$$ 

It is easy to see that

$$|\frac{zH'(z)}{H(z)} - \frac{1+r^2}{1-r^2}| \leq \frac{4r}{1-r^2}, \quad |z| = r < 1. \quad (2.2)$$

Let $a > 1/2$. It is known that [13] the disk $\{w : |w - a| < R_a\}$ is contained in the parabolic region $\{w : |w - 1| < \text{Re}w\}$ if the number $R_a$ satisfies

$$R_a = \begin{cases} a - \frac{1}{\sqrt{2a - 2}} & (\frac{1}{2} < a \leq \frac{3}{2}) \\ \frac{1}{\sqrt{2a - 2}} & (a \geq \frac{3}{2}) \end{cases}.$$ 

Let $0 \leq r \leq (4 - \sqrt{13})/3 =: \rho_0$. Then $a := (1+r^2)/(1-r^2) \leq 3/2$ for $r \leq 1/\sqrt{5} \approx 0.4472$. In particular, $a \leq 3/2$ for $0 \leq r \leq \rho_0 \approx 0.13148$. Consequently the inequality

$$\left|\frac{zH'(z)}{H(z)} - 1\right| < \text{Re}\left(\frac{zH'(z)}{H(z)}\right), \quad |z| = r < 1,$$

holds if

$$\frac{1+r^2-4r}{1-r^2} \geq \frac{1}{2},$$
or if \(3r^2 - 8r + 1 = (\rho_0 - r)(1 - 3\rho_0 r)/\rho_0 \geq 0\). As \(\rho_0 < 1/3\), this inequality is clearly satisfied for \(0 \leq r \leq \rho_0\). Also, with \(z = -\rho_0\), then

\[
\frac{|zH'(z)|}{H(z)} - 1 = \frac{1 + 4z + z^2}{1 - z^2} - 1 = \frac{4\rho_0 - 2\rho_0^2}{1 - \rho_0^2} = \frac{\rho_0^2 - 4\rho_0 + 1}{1 - \rho_0^2} = \text{Re} \left( \frac{zH'(z)}{H(z)} \right).
\]

This shows that the number \(\rho_0\) is sharp.

Define the function \(h: \mathbb{D} \to \mathbb{C}\) by \(h(z) = f(z) \ast g(z)\). Then \(h(z) = F(z) \ast G(z) \ast H(z)\) where \(F\) and \(G\) are respectively defined by \(zF'(z) = f(z)\) and \(zG'(z) = g(z)\). Since \(f, g\) are starlike, it follows that \(F\) and \(G\) are convex. Since the convolution of two convex functions is convex, \(F \ast G\) is convex. Also the function \(H(\rho_0 z)/\rho_0\) is a function in \(\mathcal{S}_\rho\) and hence \(F(z) \ast G(z) \ast H(\rho_0 z)/\rho_0\) is again in the class \(\mathcal{S}_\rho\). Equivalently, 
\[h_{\rho_0}(z) = (F \ast G \ast H)(\rho_0 z)/\rho_0\] in \(\mathcal{S}_\rho\). Thus the \(\mathcal{S}_\rho\)-radius of the function \(h\) is at least \(\rho_0\).

Consider the Koebe function \(k(z) = z/(1 - z)^2\); it is starlike and the \(\mathcal{S}_\rho\)-radius of \(k(z) \ast g(z) = (z/(1 - z)^2) \ast g(z) = zg'(z)\) is the same as the radius of uniform convexity of \(g\). Since \(\rho_0\) is the radius of uniform convexity of starlike functions, the radius is sharp. This proves the result in part (a).

We shall now prove the result in part (b). Let \(\rho_1 = (2 - 3\alpha^2)/(1 + \alpha)\). From the inequality (2.2), it follows that

\[
\text{Re} \frac{zH'(z)}{H(z)} \geq \frac{1 + r^2 - 4r}{1 - r^2} \geq \alpha
\]

for \(0 \leq |z| = r \leq \rho_1\). For \(z = -\rho_1\), then

\[
\text{Re} \frac{zH'(z)}{H(z)} = \frac{1 + \rho_1^2 - 4\rho_1}{1 - \rho_1^2} = \alpha.
\]

Since the class of starlike functions of order \(\alpha\) is also closed under convolution with convex functions, the function

\[h(\rho_1 z)/\rho_1 = (F \ast G \ast H)(\rho_1 z)/\rho_1 = F(z) \ast G(z) \ast H(\rho_1 z)/\rho_1\]

is again a starlike function of order \(\alpha\). This proves (b).

To prove (c), note that for \(0 < a < \sqrt{2}\), \(\{w : |w - a| < r_a\} \subseteq \{w : |w^2 - 1| < 1\}\) if \(r_a\) is given by

\[r_a = \begin{cases} \left(\sqrt{1 - a^2} - (1 - a^2)\right)^{1/2} & (0 < a \leq 2\sqrt{2}/3) \\ \sqrt{2} - a & (2\sqrt{2}/3 \leq a < \sqrt{2}) \end{cases} \]

Since the disk in (2.2) is centered at the point \(a := (1 + r^2)/(1 - r^2) \geq 1\), the inequality

\[
\left(\frac{zH'(z)}{H(z)}\right)^2 - 1 < 1
\]
is satisfied for $|z| = r < 1$ if
\[
\frac{4r}{1-r^2} < \sqrt{2} - \frac{1+r^2}{1-r^2},
\]
or $(\sqrt{2}+1)r^2 + 4r - (\sqrt{2} - 1) < 0$. This yields
\[
0 \leq r \leq (-2 + \sqrt{3})(\sqrt{2} - 1) =: \rho_2.
\]
For $z = \rho_2$, then
\[
\left| \left( \frac{zH'(z)}{H(z)} \right)^2 - 1 \right| = 1.
\]
The remaining part of the proof is similar to the other two parts notwithstanding the fact that the class $\mathcal{S}_L$ is closed under convolution with convex functions. However this is known even more generally for any analytic function $f$ for which $zf'(z)/f(z)$ lies in a convex domain [6]; in case of functions in the class $\mathcal{S}_L$, the convex domain is the right half of the lemniscate of Bernoulli.

The proof of part (d) is similar, with the observation that the disk $|w - a| \leq R_a$ is contained in the sector $|\arg w| \leq \frac{\pi \gamma}{2}$, $0 < \gamma \leq 1$ whenever $R_a \leq a \sin \left( \frac{\pi \gamma}{2} \right)$.

**Remark 2.3.** The proof that the $\mathcal{S}_p$-radius of the function $H$ is $\rho_0$ shows that the $\mathcal{S}_p$-radius of the (ordinary) product of a starlike function and a function with positive real part is also $\rho_0$.

**Corollary 2.4.** [4] The radius of starlikeness of convolution of two starlike functions is $2 - \sqrt{3}$.

3. Classes closed under convolution with convex functions

**Definition 3.1.** For $\alpha \geq 0$, let $\mathcal{L}(\alpha)$ be the class of analytic functions $f \in \mathcal{A}$ satisfying the condition
\[
\Re \left\{ \frac{zf'(z)}{f(z)} \left( \alpha \frac{zf''(z)}{f'(z)} + 1 \right) \right\} > -\frac{\alpha}{2}, \quad z \in \mathbb{D}.
\]

The following result is obtained.

**Theorem 3.2.** If $f \in \mathcal{L}(\alpha)$ and $\varphi \in \mathcal{C}$, then $f \ast \varphi \in \mathcal{L}(\alpha)$.

**Proof.** Let $h: \mathbb{D} \to \mathbb{C}$ be the function defined by
\[
h(z) := \frac{z + (2\alpha - 1)z^2}{(1-z)^3}.
\]
Then
\[
\frac{zf'(z)}{f(z)} \left( \alpha \frac{zf''(z)}{f'(z)} + 1 \right) = \frac{\alpha z^2 f''(z) + zf'(z)}{f(z)} = \frac{f(z) \ast h(z)}{f(z)}.
\]
Let \( F : \mathbb{D} \to \mathbb{C} \) be defined by
\[
F(z) := \frac{f(z) * h(z)}{f(z)}.
\]
The function \( F \) is clearly well-defined and analytic in \( \mathbb{D} \), with \( F(\mathbb{D}) \subseteq \{ w : \Re w > -\alpha/2 \} \). Li and Owa [3] proved that every function in the class \( L(\alpha) \) is starlike. Therefore, the function \( f * \varphi \) is starlike univalent and hence the function \( \varphi * f \) is well-defined and analytic in \( \mathbb{D} \). Also,
\[
\varphi * f \bigg|_{\mathbb{D}} \subset \text{co}(F(\mathbb{D})),
\]
or equivalently
\[
\text{Re} \left( \frac{(\varphi * f)(z)}{(f * \varphi)(z)} \right) > -\frac{\alpha}{2}, \quad z \in \mathbb{D}.
\]
This proves our result.

**Corollary 3.3.** If \( f \in L(\alpha) \), then the integral transforms \( F \) and \( G \) given by
\[
F(f(z)) = \frac{\gamma + 1}{z^\gamma} \int_0^z \zeta^{\gamma-1} f(\zeta) d\zeta, \quad \text{Re } \gamma > 0
\]
and
\[
G(f(z)) = \int_0^z \frac{f(\zeta) - f(\eta \zeta)}{\zeta - \eta \zeta} d\zeta, \quad |\eta| \leq 1, \quad \eta \neq 1
\]
are again in \( L(\alpha) \).

**Proof.** The results follow since \( F = f * \varphi_1 \) and \( G = f * \varphi_2 \) where
\[
\varphi_1(z) = \sum_{n=1}^{\infty} \frac{\gamma + 1}{\gamma + n} z^n, \quad \varphi_2(z) = \sum_{n=1}^{\infty} \frac{1 - \eta^n}{(1 - \eta) n} z^n
\]
are convex univalent functions in \( \mathbb{D} \).
Definition 3.6. For \( g \in A \) and \( \beta < 1 \), let \( R_g(\beta) \) be the class of all analytic functions \( f \in A \) satisfying the condition
\[
\text{Re} \left( \frac{(f * g)(z)}{z} \right) > \beta.
\]

We show that the class \( R_g(\beta) \) is closed under convolution with convex functions.

Theorem 3.7. If \( f \in R_g(\beta) \) and \( \varphi \in C \), then \( f * \varphi \in R_g(\beta) \).

Proof. Define the function \( F \) by
\[
F(z) = \frac{(f * g)(z)}{z}.
\]
Then
\[
\frac{(\varphi * f * g)(z)}{z} = \frac{(\varphi * (zF))(z)}{(\varphi * z)(z)} \in \sigma_c(F(D)).
\]
This shows that \( \text{Re}(\varphi * f * g)(z)/z > \beta \) and hence \( f * \varphi \in R_g(\beta) \). \( \square \)

Corollary 3.8. If \( f \in R_g(\beta) \) and the integral transforms \( F \) and \( G \) are given by (3.4) and (3.5), then \( F, G \in R_g(\beta) \).

With \( g(z) = (z + (2\alpha - 1)z^2)/(1 - z)^3 \), the class \( R_g(\beta) \) reduces to the class \( R(\alpha, \beta) \) defined below.

Definition 3.9. For \( \alpha \in \mathbb{R} \) and \( \beta < 1 \), let \( R(\alpha, \beta) \) be the class of all analytic functions \( f \in A \) satisfying the condition
\[
\text{Re} \left( f'(z) + \alpha zf''(z) \right) > \beta.
\]

Several related convolution results on \( R(\alpha, \beta) \) can be found in Ponnusamy and Singh [8]. For this class, the following results are obtained.

Corollary 3.10. If \( f \in R(\alpha, \beta) \) and \( \varphi \in C \), then \( f * \varphi \in R(\alpha, \beta) \).

Corollary 3.11. If \( f \in R(\alpha, \beta) \) and the integral transforms \( F \) and \( G \) are respectively given by (3.4) and (3.5), then \( F, G \in R(\alpha, \beta) \).

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