Abstract. A set $S$ of integers is said to be multiplicative if for every pair $m$ and $n$ of coprime integers we have that $mn$ is in $S$ if both $m$ and $n$ are in $S$. Both Landau and Ramanujan gave approximations to $S(x)$, the number of $n \leq x$ that are in $S$, for specific choices of $S$. The asymptotical precision of their respective approximations are being compared and related to Euler-Kronecker constants, a generalization of Euler’s constant $\gamma = 0.57721566 \ldots$.

This paper claims little originality, its aim is to give a survey on the literature related to this theme with an emphasis on the contributions of the author (and his coauthors).

2000 Mathematics Subject Classification. 11N37; 11Y60

1. Introduction

To every prime $p$ we associate a set $E(p)$ of positive allowed exponents. Thus $E(p)$ is a subset of $\mathbb{N}$. We consider the set $S$ of integers consisting of 1 and all integers of the form $n = \prod_i p_i^{e_i}$ with $e_i \in E(p_i)$. Note that this set is multiplicative, i.e., if $m$ and $n$ are coprime integers then $mn$ is in $S$ if both $m$ and $n$ are in $S$. It is easy to see that in this way we obtain all multiplicative sets of natural numbers. As an example, let us consider the case where $E(p)$ consists of the positive even integers if $p \equiv 3 \pmod{4}$ and $E(p) = \mathbb{N}$ for the other primes.

The set $S_B$ obtained in this way can be described in another way. By the well-known result that every positive integer can be written as a sum of two squares iff every prime divisor $p$ of $n$ of the form $p \equiv 3 \pmod{4}$ occurs to an even exponent, we see that $S_B$ is the set of positive integers that can be written as a sum of two integer squares.

In this note we are interested in the counting function associated to $S$, $S(x)$, which counts the number of $n \leq x$ that are in $S$. By $\pi_S(x)$ we denote the number of primes $p \leq x$ that are in $S$. We will only consider $S$ with the property that $\pi_S(x)$ can be well-approximated by $\delta \pi(x)$ with $\delta > 0$ real and $\pi(x)$ the prime counting function (thus $\pi(x) = \sum_{p \leq x} 1$). Recall that the Prime Number Theorem states that asymptotically $\pi(x) \sim x / \log x$. Gauss as a teenager conjectured that the logarithmic integral, $\text{Li}(x)$, defined as $\int_2^x dt / \log t$ gives a much better approximation to $\pi(x)$. Indeed, it is now known that, for any $r > 0$ we have $\pi(x) = \text{Li}(x) + O(x^{\log^{-r} x})$.

On the other hand, the result that $\pi(x) = x / \log x + O(x \log^{-r} x)$, is false for $r > 2$. In this note two types of approximation of $\pi_S(x)$ by $\delta \pi(x)$ play an important role. We say $S$ satisfies Condition A if, asymptotically,

$$\pi_S(x) \sim \frac{x}{\log x} \delta. \tag{1}$$

We say that $S$ satisfies Condition B if there are some fixed positive numbers $\delta$ and $\rho$ such that asymptotically

$$\pi_S(x) = \delta \text{Li}(x) + O \left( \frac{x}{\log^{2}{\rho} x} \right). \tag{2}$$

The following result is a special case of a result of Wirsing [36], with a reformulation following Finch et al. [9, p. 2732]. As usual $\Gamma$ will denote the gamma function. By $\chi_S$ we denote the characteristic function of $S$, that is we put $\chi_S(n) = 1$ if $n$ is in $S$ and zero otherwise.

Theorem 1. Let $S$ be a multiplicative set satisfying Condition A, then

$$S(x) \sim C_0(S) x \log^{\delta-1} x,$$

where

$$C_0(S) = \frac{1}{\Gamma(\delta)} \lim_{P \to \infty} \prod_{p < P} \left(1 + \frac{\chi_S(p)}{p} + \frac{\chi_S(p^2)}{p^2} + \cdots \right) \left(1 - \frac{1}{p}\right)^\delta,$$

converges and hence is positive.
In case $S = S_B$ we have $\delta = 1/2$ by Dirichlet’s prime number theorem for arithmetic progressions. Recall that for fixed $r > 0$ this theorem states that

$$\pi(x; d, a) := \sum_{p \leq x, p \equiv a (\mod d)} 1 = \frac{\text{Li}(x)}{\varphi(d)} + O\left(\frac{x}{\log^r x}\right).$$

Theorem 1 thus gives that, asymptotically, $S_B(x) \sim C_0(S_B)x/\sqrt{\log x}$, a result derived in 1908 by Edmund Landau. Ramanujan, in his first letter to Hardy (1913), wrote in our notation that

$$S_B(x) = C_0(S_B) \int_2^x \frac{dt}{\log t} + \theta(x),$$

with $\theta(x)$ very small. In reply to Hardy’s question what ‘very small’ is in this context Ramanujan wrote back

$$\log^\delta_x = = \text{large}$$

leads us to the following definition.

**Definition 1.** Let $S$ be a multiplicative set such that $\pi_S(x) \sim \delta x/\log x$ for some $\delta > 0$. If for all $x$ sufficiently large

$$|S(x) - C_0(S)x \log^{\delta - 1} x| < |S(x) - C_0(S) \int_2^x \log^{\delta - 1} dt|,$$

for every $x$ sufficiently large, we say that the Landau approximation is better than the Ramanujan approximation. If the reverse inequality holds for every $x$ sufficiently large, we say that the Ramanujan approximation is better than the Landau approximation.

We denote the formal Dirichlet series $\sum_{n=1}^\infty n^{-s}$ associated to $S$ by $L_S(s)$. For $\text{Re}(s) > 1$ it converges. If

$$\gamma_S := \lim_{s \to 1+0} \left( \frac{L_S(s)}{\zeta(s)} + \frac{\delta}{s-1} \right)$$

exists, we say that $S$ has Euler-Kronecker constant $\gamma_S$. In case $S$ consists of all positive integers we have $L_S(s) = \zeta(s)$ and it is well known that

$$\lim_{s \to 1+0} \left( \frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1} \right) = \gamma.$$ (5)

If the multiplicative set $S$ satisfies condition B, then it can be shown that $\gamma_S$ exists. Indeed, we have the following result.

**Theorem 2 [19].** If the multiplicative set $S$ satisfies Condition B, then

$$S(x) = C_0(S)x \log^{\delta - 1} x \left( 1 + (1 + o(1)) \frac{C_1(S)}{\log x} \right),$$

as $x \to \infty$.

where $C_1(S) = (1 - \delta)(1 - \gamma_S)$.

**Corollary 1.** Suppose that $S$ is multiplicative and satisfies Condition B. If $\gamma_S < 1/2$, then the Ramanujan approximation is asymptotically better than the Landau one. If $\gamma_S > 1/2$ it is the other way around.

The corollary follows on noting that by partial integration we have

$$\int_2^x \log^{\delta - 1} dt = x \log^{\delta - 1} x \left( 1 + \frac{1 - \delta}{\log x} + O\left(\frac{1}{\log^2 x}\right) \right).$$ (6)

On comparing (6) with Theorem 2 we see Ramanujan’s claim (3), if true, implies $\gamma_S = 0$.

A special, but common case, is where the primes in the set $S$ are, with finitely many exceptions, precisely those in a finite union of arithmetic progressions, that is, there exists a modulus $d$ and integers $a_1, \ldots, a_s$ such that for all sufficiently large primes $p$ we have $p \equiv a_i (\mod d)$ for some $1 \leq i \leq s$. Indeed, all examples we consider in this paper belong to this special case.) Under this assumption it can be shown, see Serre [28], that $S(x)$ has an asymptotic expansion in the sense of Poincaré, that is, for every integer $m \geq 1$ we have

$$S(x) = C_0(S)x \log^{\delta - 1} x \left( 1 + \frac{C_1(S)}{\log x} + \frac{C_2(S)}{\log^2 x} + \cdots \right.$$

$$+ \frac{C_m(S)}{\log^m x} + O\left(\frac{1}{\log^{m+1} x}\right),$$ (7)

where the implicit error term may depend on both $m$ and $S$. In particular $S_b(x)$ has an expansion of the form (7) (see, e.g., Hardy [12, p. 63] for a proof).

2. On the Numerical Evaluation of $\gamma_S$

We discuss various ways of numerically approximating $\gamma_S$.

A few of these approaches involve a generalization of the von Mangoldt function $\Lambda(n)$ (for more details see Section 2.2 of [20]).

We define $\Lambda_S(n)$ implicitly by

$$\frac{L_S(s)}{L_S(2)} = \sum_{n=1}^\infty \frac{\Lambda_S(n)}{n^s}.$$ (8)

As an example let us compute $\Lambda_S(n)$ in case $S = \mathbb{N}$. Since

$$L_S(s) = \xi(s) = \prod_p \left( 1 - \frac{1}{p^s} \right)^{-1},$$
we obtain \( \log \zeta(s) = -\sum_p \log(1 - p^{-s}) \) and hence
\[
-\frac{L'_S(s)}{L_S(s)} = -\frac{\zeta'(s)}{\zeta(s)} = \sum_p \frac{\log p}{p^s - 1}.
\]

We infer that \( \Lambda_S(n) = \Lambda(n) \), the von Mangoldt function. Recall that
\[
\Lambda(n) = \begin{cases} 
\log p & \text{if } n = p^r; \\
0 & \text{otherwise.}
\end{cases}
\]

In case \( S \) is a multiplicative semigroup generated by \( q_1, q_2, \ldots \), we have
\[
L_S(s) = \prod_i \left(1 - \frac{1}{q_i^s}\right)^{-1},
\]
and we find
\[
\Lambda_S(n) = \begin{cases} 
\log q_i & \text{if } n = q_i^r; \\
0 & \text{otherwise.}
\end{cases}
\]

Note that \( S_B \) is a multiplicative semigroup. It is generated by 2, the primes \( p \equiv 1 (\mod 4) \) and the squares of the primes \( p \equiv 3 (\mod 4) \).

For a more general multiplicative set \( \Lambda_S(n) \) can become more difficult in nature as we will now argue. We claim that (8) gives rise to the identity
\[
\chi_S(n) \log n = \sum_{d|n} \chi_S \left(\frac{n}{d}\right) \Lambda_S(d). \tag{9}
\]
In the case \( S = \mathbb{N} \), e.g., we obtain \( \log n = \sum_{d|n} \Lambda(d) \). In order to derive (9) we use the observation that if \( F(s) = \sum f(n)n^{-s} \), \( G(s) = \sum g(n)n^{-s} \) and \( F(s)G(s) = \mathcal{H}(s) = \sum h(n)n^{-s} \) are formal Dirichlet series, then \( h \) is the Dirichlet convolution of \( f \) and \( g \), i.e., \( h(n) = (f * g)(n) = \sum_{d|n} f(d)g(n/d) \). By an argument similar to the one that led us to the von Mangoldt function, one sees that \( \Lambda_S(n) = 0 \) in case \( n \) is not a prime power. Thus we can rewrite (9) as
\[
\chi_S(n) \log n = \sum_{p^r|n} \chi_S \left(\frac{n}{d}\right) \Lambda_S(d). \tag{10}
\]
By induction one then finds that \( \Lambda_S(p^e) = c_S(p^e) \log p \), where \( c_S(p) = \chi_S(p) \) and \( c_S(p^e) \) is defined recursively for \( e > 1 \) by
\[
c_S(p^e) = e c_S(p^e) - \sum_{j=1}^{e-1} c_S(p^j) \chi_S(p^{e-j}).
\]

Also a more closed expression for \( \Lambda_S(n) \) can be given ([20, Proposition 13]), namely
\[
\Lambda_S(n) = e \log p \sum_{m=1}^e \left(-1\right)^{m-1} \frac{m}{m} 
\times \sum_{k_1 + k_2 + \ldots + k_e = m} \chi_S(p^{k_1}) \chi_S(p^{k_2}) \ldots \chi_S(p^{k_e}),
\]
if \( n = p^e \) for some \( e \geq 1 \) and \( \Lambda_S(n) = 0 \) otherwise, or alternatively \( \Lambda_S(n) = W e \log p \), where
\[
W = \sum_{l_1 + 2l_2 + \ldots + el_e = e} \frac{(-1)^{l_1 + \ldots + l_e}}{l_1!l_2!\ldots l_e!} \left(\frac{l_1 + l_2 + \ldots + l_e}{l_1!^2l_2!^2\ldots l_e!^2}\right),
\]
if \( n = p^e \) and \( \Lambda_S(n) = 0 \) otherwise, where the \( k_i \) run through the natural numbers and the \( l_j \) through the non-negative integers.

Now that we can compute \( \Lambda_S(n) \) we are ready for some formulae expressing \( \gamma_S \) in terms of this function.

**Theorem 3.** Suppose that \( S \) is a multiplicative set satisfying Condition B. Then
\[
\sum_{n \leq x} \Lambda_S(n) = e \delta \log x - \gamma_S + O\left(\frac{1}{\log^a x}\right).
\]
Moreover, we have
\[
\gamma_S = -\delta \gamma + \sum_{n=1}^{\infty} \frac{\delta - \Lambda_S(n)}{n}.
\]
In case \( S \) is a semigroup generated by \( q_1, q_2, \ldots \), then one has
\[
\gamma_S = \lim_{x \to \infty} \left(\delta \log x - \sum_{q_i \leq x} \log q_i - \sum_{q_i > x} \frac{1}{n} \right).
\]
The second formula given in Theorem 3 easily follows from the first on invoking the classical definition of \( \gamma \):
\[
\gamma = \lim_{x \to \infty} \left(\sum_{n \leq x} \frac{1}{n} - \log x\right).
\]

Theorem 3 is quite suitable for getting an approximative value of \( \gamma_S \). The formulae given there, however, do not allow one to compute \( \gamma_S \) with a prescribed numerical precision. For doing that another approach is needed, the idea of which is to relate the generating series \( L_S(s) \) to \( \zeta(s) \) and then take the logarithmic derivative. We illustrate this in Section 4, by showing how \( \gamma_{S_B} \) (defined in that section) can be computed with high numerical precision.
3. Non-Divisibility of Multiplicative Arithmetic Functions

Given a multiplicative arithmetic function \( f \) taking only integer values, it is an almost immediate observation that, with \( q \) a prime, the set \( S_{f,q} := \{ n : q \nmid f(n) \} \) is multiplicative.

3.1 Non-divisibility of Ramanujan’s \( \tau \)

In his so-called ‘unpublished’ manuscript on the partition and tau functions [1,3], Ramanujan considers the counting function of \( S_{\tau,q} \), where \( q \in \{3, 5, 7, 23, 691\} \) and \( \tau \) is the Ramanujan \( \tau \)-function. Ramanujan’s \( \tau \)-function is defined as the coefficients of the power series in \( q \):

\[
\Delta := q \prod_{m=1}^{\infty} (1 - q^m)^{24} = \sum_{n=1}^{\infty} \tau(n)q^n.
\]

After setting \( q = e^{2\pi i z} \), the function \( \Delta(z) \) is the unique normalized cusp form of weight 12 for the full modular group \( \text{SL}_2(\mathbb{Z}) \). It turns out that \( \tau \) is a multiplicative function and hence the set \( S_{\tau,q} \) is multiplicative. Given any such \( S_{\tau,q} \), Ramanujan denotes \( \chi_{S_{\tau,q}}(n) \) by \( t_q \). He then typically writes: “It is easy to prove by quite elementary methods that \( \sum_{k=1}^{n} t_q = o(n) \). It can be shown by transcendental methods that

\[
\sum_{k=1}^{n} t_q \sim \frac{Cn}{\log^r n}, \quad (11)
\]

and

\[
\sum_{k=1}^{n} t_q = C \int_{2}^{n} \frac{dx}{\log^r x} + O\left( \frac{n}{\log^r n} \right), \quad (12)
\]

where \( r \) is any positive number. Ramanujan claims that \( \delta_3 = \delta_7 = \delta_{23} = 1/2, \delta_5 = 1/4 \) and \( \delta_{691} = 1/690 \). Except for \( q = 5 \) and \( q = 691 \) Ramanujan also writes down an Euler product for \( C \). These are correct, except for a minor omission he made in case \( q = 23 \).

Theorem 4 [17]. For \( q \in \{3, 5, 7, 23, 691\} \) we have \( \gamma_{S_{\tau,q}} \neq 0 \) and thus Ramanujan’s claim (12) is false for \( r > 2 \).

The reader might wonder why this specific small set of \( q \). The answer is that in these cases Ramanujan established easy congruences such as

\[
\tau(n) \equiv \sum_{d|n} d^{11} \pmod{691}
\]

that allow one to easily describe the non-divisibility of \( \tau(n) \) for these \( q \). Serre, see [34], has shown that for every odd prime \( q \) a formula of type (11) exists, although no simple congruences as above exist. This result requires quite sophisticated tools, e.g., the theory of \( l \)-adic representations. The question that arises is whether \( \gamma_{S_{\tau,q}} \) exists for every odd \( q \) and if yes, to compute it with enough numerical precision as to determine whether it is zero or not and to be able to tell whether the Landau or the Ramanujan approximation is better.

3.2 Non-divisibility of Euler’s totient function \( \varphi \)

Spearman and Williams [32] determined the asymptotic behaviour of \( S_{\varphi,q}(x) \). Here invariants from the cyclotomic field \( \mathbb{Q}(\zeta_q) \) come into play. The mathematical connection with cyclotomic fields is not very direct in [32]. However, this connection can be made and in this way the results of Spearman and Williams can then be re-derived in a rather straightforward way, see [10,19]. Recall that the Extended Riemann Hypothesis (ERH) says that the Riemann Hypothesis holds true for every Dirichlet \( L \)-series \( L(s, \chi) \).

Theorem 5 [10]. For \( q \leq 67 \) we have \( 1/2 > \gamma_{S_{\varphi,q}} > 0 \). For \( q > 67 \) we have \( \gamma_{S_{\varphi,q}} > 1/2 \). Furthermore we have \( \gamma_{S_{\varphi,q}} = \gamma + O(\log^3 q/\sqrt{q}), \) unconditionally with an effective constant, \( \gamma_{S_{\varphi,q}} = \gamma + O(q^{-1}), \) unconditionally with an ineffective constant and \( \gamma_{S_{\varphi,q}} = \gamma + O((\log q)(\log \log q)/q) \) if ERH holds true.

The explicit inequalities in this result were first proved by the author [19], who established them assuming ERH. Note that the result shows that Landau wins over Ramanujan for every prime \( q \geq 71 \).

Given a number field \( K \), the Euler-Kronecker constant \( \mathcal{E}_K \) of the number field \( K \) is defined as

\[
\mathcal{E}_K = \lim_{x \to \infty} \left( \frac{\zeta_K(s)}{\zeta(s)} - 1 \right),
\]

where \( \zeta_K(s) \) denotes the Dedekind zeta-function of \( K \). Given a prime \( p \neq q \), let \( f_p \) the smallest positive integer such that \( p^{f_p} \equiv 1 \pmod{q} \). Put

\[
S(q) = \sum_{p \neq q, f_p \geq 2} \frac{\log p}{p^{f_p} - 1}.
\]

We have

\[
\gamma_{S_{\varphi,q}} = \gamma - \frac{(3 - q) \log q}{(q - 1)^2(q + 1)} - S(q) - \frac{2 \mathcal{E}_{\mathbb{Q}(\zeta_q)}}{q - 1}, \quad (13)
\]
The total arclength of the lemniscate $r$ constant.

M( where by 2 diary) that starting values

The value set of a positive definite integral binary

Theorem 6. The value set of a positive definite integral binary

4. Some Euler-Kronecker Constants Related
to Binary Quadratic Forms

Hardy [12, p. 9, p. 63] was under the misapprehension that for $S_k$ Landau’s approximation is better. However, he based himself on a computation of his student Gertrude Stanley [33] that turned out to be incorrect. Shanks proved that

$$\gamma S_n = \frac{\gamma}{2} + \frac{1}{2} L' \left( \chi_{-4} \right) - \frac{\log 2}{2} - \sum_{p \equiv 3 \text{mod 4}} \frac{\log p}{p^2 - 1}. \quad (14)$$

Various mathematicians independently discovered the result that

$$L' \left( \chi_{-4} \right) = \log(M(1, \sqrt{2})^2 e^{\beta} / 2),$$

where $M(1, \sqrt{2})$ denotes the limiting value of Lagrange’s AGM algorithm $a_{n+1} = (a_n + b_n)/2, b_{n+1} = \sqrt{a_n b_n}$ with starting values $a_0 = 1$ and $b_0 = \sqrt{2}$. Gauss showed (in his diary) that

$$\frac{1}{M(1, \sqrt{2})} = \frac{2}{\pi} \int_0^1 \frac{dx}{\sqrt{1-x^4}}.$$

The total arclength of the lemniscate $r_2 = \cos(2\theta)$ is given by $2l$, where $L = \pi / M(1, \sqrt{2})$ is the so-called lemniscate constant.

Shanks used these formulæ to show that $\gamma S_n = -0.1638973186345 \ldots \neq 0$, thus establishing the falsity of Ramanujan’s claim (3). Since $\gamma S_n < 1/2$, it follows by Corollary 1 that actually the Ramanujan approximation is better.

A natural question is to determine the primitive binary quadratic forms $f(X, Y) = aX^2 + bXY + cY^2$ of negative discriminant for which the integers represented form a multiplicative set. This does not seem to be known. However, in the more restrictive case where we require the multiplicative set to be also a semigroup the answer is known, see Earnest and Fitzgerald [7].

**Theorem 6.** The value set of a positive definite integral binary quadratic form forms a semigroup if and only if it is in the principal class, i.e. represents 1, or has order 3 (under Gauss composition).

In the former case, the set of represented integers is just the set of norms from the order $\mathcal{O}_D$, which is multiplicative.

In the latter case, the smallest example comes from the forms of discriminant $-23$, for which the class group is cyclic of order 3: the primes $p$ are partitioned into those of the form $X^2 - XY + 6Y^2$ and those of the form $2X^2 \pm XY + 3Y^2$.

Although the integers represented by $f(X, Y)$ do not in general form a multiplicative set, the associated set $I_f$ of integers represented by $f$, always satisfies the same type of asymptotic, namely we have

$$I_f(x) \sim C_f \frac{x}{\sqrt{\log x}}.$$

This result is due to Paul Bernays [2], of fame in logic, who did his PhD thesis with Landau. Since his work was not published in a mathematical journal it got forgotten and later partially rediscovered by mathematicians such as James and Pall. For a brief description of the proof approach of Bernays see Brink et al. [4].

We like to point out that in general the estimate

$$I_f(x) = C_f \frac{x}{\sqrt{\log x}} \left( 1 + (1 + o(1)) \frac{C_f'}{\log x} \right)$$

does not hold. For example, for $f(X, Y) = X^2 + 14Y^2$, see Shanks and Schmid [31].

Bernays did not compute $C_f$, this was only done much later and required the combined effort of various mathematicians. The author and Osburn [21] combined these results to show that of all the two dimensional lattices of covolume 1, the hexagonal lattice has the fewest distances. Earlier Conway and Sloane [6] had identified the lattices with fewest distances in dimensions 3 to 8, also relying on the work of many other mathematicians.

In the special case where $f = X^2 + nY^2$, a remark in a paper of Shanks seemed to suggest that he thought $C_f$ would be maximal in case $n = 2$. However, the maximum does not occur for $n = 2$, see Brink et al. [4].

In estimating $I_f(x)$, the first step is to count $B_D(x)$. Given a discriminant $D \leq -3$ we let $B_D(x)$ count the number of integers $n \leq x$ that are coprime to $D$ and can be represented by some primitive quadratic integral form of discriminant $D$. The integers so represented are known, see e.g. James [15], to form a multiplicative semigroup, $S_D$, generated by the primes $p$ with $(D / p) = 1$ and the squares of the primes $q$ with $(D / q) = -1$. James [15] showed that we have...
\[
B_D(x) = C(S_D) \frac{x}{\sqrt{\log x}} + O \left( \frac{x}{\log x} \right).
\]

An easier proof, following closely the ideas employed by Rieger [25], was given by Williams [35]. The set of primes in \( S_D \) has density \( \delta = 1/2 \). By the law of quadratic reciprocity the set of primes \( p \) satisfying \( \left( \frac{D}{p} \right) = 1 \) is, with finitely many exceptions, precisely a union of arithmetic progressions. It thus follows that Condition B is satisfied and, moreover, that for every integer \( m \geq 1 \), we have an expansion of the form

\[
B_D(x) = C(S_D) \frac{x}{\sqrt{\log x}} \times \left( 1 + \frac{b_1}{\log x} + \frac{b_2}{\log^2 x} + \cdots + O \left( \frac{1}{\log^m x} \right) \right).
\]

By Theorem 2 and Theorem 3 we infer that \( b_1 = (1 - \gamma_{S_D})/2 \), with

\[
\gamma_{S_D} = \lim_{x \to \infty} \left( \frac{\log x}{2} - \sum_{p \leq x} \frac{\log p}{p-1} \right) - \sum_{(\frac{D}{p}) = -1} \frac{2 \log p}{p^2 - 1}.
\]

As remarked earlier, in order to compute \( \gamma_{S_D} \) with some numerical precision the above formula is not suitable and another route has to be taken.

**Proposition 1** [15]. We have, for \( \text{Re}(s) > 1 \),

\[
L_{S_D}(s)^2 = \zeta(s) L(s, \chi_D) \prod_{(\frac{D}{p}) = -1} (1 - p^{-2s})^{-1} \prod_{p | D}(1 - p^{-s}).
\]

**Proof.** Noting that

\[
L_{S_D}(s) = \prod_{(\frac{D}{p}) = 1} (1 - p^{-s})^{-1} \prod_{(\frac{D}{p}) = -1} (1 - p^{-2s})^{-1},
\]

and

\[
L(s, \chi_D) = \prod_{(\frac{D}{p}) = 1} (1 - p^{-s})^{-1} \prod_{(\frac{D}{p}) = -1} (1 + p^{-s})^{-1},
\]

the proof follows on comparing Euler factors on both sides.

**Proposition 2.** We have

\[
2\gamma_{S_D} = \gamma + \frac{L'}{L}(1, \chi_D) - \sum_{(\frac{D}{p}) = -1} \frac{2 \log p}{p^2 - 1} + \sum_{p | D} \frac{\log p}{p - 1}.
\]

**Proof.** Follows on logarithmically differentiating the expression for \( L_{S_D}(s)^2 \) given in Proposition 1, invoking (5) and recalling that \( L(1, \chi_D) \neq 0 \).

The latter result together with \( b_1 = (1 - \gamma_{S_D})/2 \) leads to a formula first proved by Heupel [13] in a different way.

The first sum appearing in Proposition 2 can be evaluated with high numerical precision by using the identity

\[
\sum_{(\frac{D}{p}) = -1} \frac{2 \log p}{p^2 - 1} = \sum_{k=1}^{\infty} \left( \frac{L'}{L}(2k, \chi_D) - \frac{\zeta'}{\zeta}(2k) - \sum_{p | D} \frac{\log p}{p^{2k} - 1} \right).
\]

This identity in case \( D = -3 \) was established in [18, p. 436]. The proof given there is easily generalized. An alternative proof follows on combining Proposition 3 with Proposition 4.

**Proposition 3.** We have

\[
\sum_{p} \frac{(\frac{D}{p}) \log p}{p - 1} = -\frac{L'}{L}(1, \chi_D) + \sum_{k=1}^{\infty} \left( -\frac{L'}{L}(2k, \chi_D) + \frac{\zeta'}{\zeta}(2k) + \sum_{p | D} \frac{\log p}{p^{2k} - 1} \right).
\]

**Proof.** This is Lemma 12 in Cilleruelo [5].

**Proposition 4.** We have

\[
-\sum_{p} \frac{(\frac{D}{p}) \log p}{p - 1} = \frac{L'}{L}(1, \chi_D) + \sum_{(\frac{D}{p}) = -1} \frac{2 \log p}{p^2 - 1}.
\]

**Proof.** Put \( G_d(s) = \prod_{p} (1 - p^{-s})^{(D/p)} \). We have

\[
\frac{1}{G_d(s)} = L(s, \chi_D) \prod_{(\frac{D}{p}) = -1} (1 - p^{-2s}).
\]

The result then follows on logarithmic differentiation of both sides of the identity and the fact that \( L(1, \chi_D) \neq 0 \).

The terms in (15) can be calculated with MAGMA with high precision and the series involved converge very fast. Cilleruelo [5] claims that

\[
\sum_{k=1}^{\infty} \frac{L'(2k, \chi_D)}{L(2k, \chi_D)} = \sum_{k=1}^{6} \frac{L'(2k, \chi_D)}{L(2k, \chi_D)} + \text{Error}, \text{Error} \leq 10^{-40}.
\]

We will now re-derive Shanks’s result (14). Since there is only one primitive quadratic form of discriminant \(-4\), we see
that $S_{-4}$ is precisely the set of odd integers that can be written as a sum of two squares. If $m$ is an odd integer that can be written as a sum of two squares, then so can $2^em$ with $e \geq 0$ arbitrary. It follows that $L_{S_{m}}(s) = (1 - 2^{-s})^{-1}L_{S_{-4}}(s)$ and hence $\gamma_{S_m} = \gamma_{S_{-4}} - \log 2$. On invoking Proposition 2 one then finds the identity (14).

5. Integers Composed Only of Primes in a Prescribed Arithmetic Progression

Consider an arithmetic progression having infinitely many primes in it, that is consider the progression $a, a + d, a + 2d, \ldots$ with $a$ and $d$ coprime. Let $S_{d,a}$ be the multiplicative set of integers composed only of primes $p \equiv a (mod \, d)$. Here we will only consider the simple case where $a = 1$ and $d = q$ is a prime number. This problem is very closely related to that in Section 3.2. One has $L_{S_{q}}(s) = (1 + q^{-s}) \prod_{p \equiv 1 (mod \, q)} (1 - p^{-s})^{-1}$. Since $L_{S_{q}}(s) = \prod_{p \equiv 1 (mod \, q)} (1 - p^{-s})^{-1}$, we then infer that

$$L_{S_{q}}(s)L_{S_{p}}(s) = \gamma(s)(1 - q^{-s})$$

and hence

$$\gamma_{S_{q}} = \gamma_{S_{p}} + \frac{2 \log q}{q^2 - 1} = \frac{\log q}{(q - 1)^2} + \frac{\xi_{q}(q)}{\zeta(q)}, \quad (16)$$

where the latter equality follows by identity (13). By Theorem 5, (16) and the Table in Ford et al. [10], we then arrive after some easy analysis at the following result.

**Theorem 7.** For $q \leq 7$ we have $\gamma_{S_{q}} < 0.5247$. For $q > 7$ we have $\gamma_{S_{q}} < -0.2862$. Furthermore we have $\gamma_{S_{q}} = O(\log^2 q/\sqrt{q})$, unconditionally with an effective constant, $\gamma_{S_{q}} = O(q^{s-1})$, unconditionally with an ineffective constant and $\gamma_{S_{q}} = O((\log q)(\log \log q)/q)$ if ERH holds true.

6. Multiplicative Set Races

Given two multiplicative sets $S_1$ and $S_2$, one can wonder whether for every $x \geq 0$ we have $S_1(x) \geq S_2(x)$. We give an example showing that this question is not as far-fetched as one might think at first sight. Schmutz Schaller [27, p. 201], motivated by considerations from hyperbolic geometry, conjectured that the hexagonal lattice is better than the square lattice, by which he means that $S_{h}(x) \geq S_{s}(x)$ for every $x$, where $S_{h}$ is the set of squared distances occurring in the hexagonal lattices, that is the integers represented by the quadratic form $X^2 + XY + Y^2$. It is well-known that the numbers represented by this form are the integers generated by the primes $p \equiv 1 (mod \, 3)$, 3 and the numbers $p^2$ with $p \equiv 2 (mod \, 3)$. Thus $S_{h}$ is a multiplicative set. If $0 < h_1 < h_2 < \cdots$ are the elements in ascending order in $S_{h}$ and $0 < q_1 < q_2 < \cdots$ the elements in ascending order in $S_{b}$, then the conjecture can also be formulated (as Schmutz Schaller did) as $q_j \leq h_j$ for every $j \geq 1$. Asymptotically one easily finds that

$$S_{b}(x) \sim C_{0}(S_{b}) \frac{x}{\sqrt{\log x}}, \quad S_{h}(x) \sim C_{0}(S_{h}) \frac{x}{\sqrt{\log x}},$$

with $C_{0}(S_{b}) \approx 0.764$ the Landau–Ramanujan constant (see Finch [8, Section 2.3]) and $C_{0}(S_{h}) \approx 0.639 \cdots$. It is thus clear that asymptotically the conjecture holds true. However, if one wishes to make the above estimates effective, matters become much more complicated. Nonetheless, the author, with computational help of H. te Riele, managed to establish the conjecture of Schmutz Schaller.

**Theorem 8 [22].** If $S_{b}$ races against $S_{h}$, $S_{b}$ is permanently ahead, that is, we have $S_{b}(x) \geq S_{h}(x)$ for every $x \geq 0$.

Many of the ideas used to establish the above result were first developed in [18]. There some other multiplicative set races where considered. Given coprime positive integers $a$ and $d$, let $S_{d,a}$ be the multiplicative set of integers composed only of primes $p \equiv a (mod \, d)$. The author established the following result as a precursor to Theorem 8.

**Theorem 9 [18].** For every $x \geq 0$ we have $S_{d,1}(x) \geq S'_{3,1}(x)$, $S_{4,3}(x) \geq S'_{3,1}(x)$, $S'_{3,2}(x) \geq S'_{4,1}(x)$ and $S'_{4,3} \geq S'_{4,1}(x)$.

We like to point out that in every race mentioned in the latter result, the associated prime number races have no ultimate winner. For example, already Littlewood [16] in 1914 showed that $\pi_{S_{d,1}}(x) - \pi_{S_{d,1}}(x)$ has infinitely many sign changes. Note that trivially if $\pi_{S_{d,1}}(x) \geq \pi_{S_{d,1}}(x)$ for every $x \geq 0$, then $S'_{d,1}(x) \geq S'_{d,1}(x)$ for every $x \geq 0$.

7. Exercises

Exercise 1. The non-hypotenuse numbers \( n = 1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 14, 16, \ldots \) are those natural numbers for which there is no solution of \( n^2 = u^2 + v^2 \) with \( u > v > 0 \) integers. The set \( S_{NH} \) of non-hypotenuse numbers forms a multiplicative set that is generated by 2 and all the primes \( p \equiv 3 \pmod{4} \). Show that \( L_{NH}(s) = L_{S_n}(s)/L(s, \chi_{-4}) \) and hence

\[
2\gamma_{NH} = 2\gamma_{S_n} - \frac{2L'(1, \chi_{-4})}{L(1, \chi_{-4})} = \gamma - \log 2 + \sum_{p>2} \left(\frac{-1}{p}\right) \frac{\log p}{p-1}.
\]

Remark. Put \( f(x) = x^2 + 1 \). Cilleruelo [5] showed that, as \( n \) tends to infinity,

\[
\log \text{l.c.m.}(f(1), \ldots, f(n)) = n \log n + Jn + o(n),
\]

with

\[
J = \gamma - 1 - \frac{\log 2}{2} - \sum_{p>3} \left(\frac{-1}{p}\right) \frac{\log p}{p-1} = -0.0662756342\ldots
\]

We have \( J = 2\gamma - 1 - \frac{1}{2} \log 2 - 2\gamma_{NH} \).

Recently the error term \( o(n) \) has been improved by Ruè et al. [26] to

\[
O\left(\frac{n}{\log^{3/2+\epsilon} n}\right),
\]

with \( \epsilon > 0 \).

Exercise 2. Let \( S'_D \) be the semigroup generated by the primes \( p \) with \( \left(\frac{D}{p}\right) = -1 \). It is easy to see that \( L_{S'_n}(s)^2 = L_{S_n}(s)^2 L(s, \chi_D)^{-2} \) and hence, by Proposition 2, we obtain

\[
2\gamma_{S'_n} = 2\gamma_{S_n} - \frac{2L'(1, \chi_D)}{L(1, \chi_D)} = \gamma - \frac{L'(1, \chi_D)}{L(1, \chi_D)} - \sum_{p=1}^{\infty} \frac{2 \log p}{p^2 - 1} + \sum_{p \mid D} \frac{\log p}{p-1}.
\]

Acknowledgement

I like to thank Andrew Earnest and John Voight for helpful information concerning quadratic forms having a value set that is multiplicative, and Ana Zumalacarregui for sending me [26]. Bruce Berndt kindly commented on an earlier version and sent me an early version of [1].

References


Dedekind’s Proof of Euler’s Reflection Formula via ODEs

Gopala Krishna Srinivasan

Department of Mathematics, Indian Institute of Technology Bombay, Mumbai 400 076
E-mail: gopal@math.iitb.ac.in

Among the higher transcendental functions, Euler’s gamma function enjoys the privilege of being most popularly studied. The text [4] contains a short but elegant account of it in the real domain via its characterization due to Bohr and Mollerup. Among its properties the most striking is the reflection formula of Euler:

$$\Gamma(a)\Gamma(1-a) = \pi \csc \pi a, \quad 0 < a < 1,$$

which is usually proved using Cauchy’s residue theorem or the infinite product expansion for \(\sin \pi x\). Richard Dedekind wrote his PhD dissertation ([2], [3]) on the gamma function under the supervision of C. F. Gauss in which he has given an interesting but elementary real variables proof of (1) by solving an initial value problem for a second order ordinary differential equation (7). Surprisingly, this proof seems to have escaped notice completely though it appears among the many exercises in [1]. The purpose of this brief note is to popularize Dedekind’s proof. For historical details and other approaches to the gamma function see [5] and the references therein. The exchange of integrals and differentiation under the integrals carried out here fall within the scope of Fubini’s theorem and Lebesgue’s dominated convergence theorem.

Recalling the classical beta-gamma identity ([4], p. 193), we denote \(B(a, 1-a)\) by \(B(a)\) and (1) is equivalent to proving

$$B(a) = \int_0^\infty \frac{x^{a-1}}{1+x}dx = \pi \csc \pi a. \quad (2)$$

The proof is broken up into easy lemmas, the first of which recasts (2) in a symmetric form:

**Lemma 1.**

(i) \(B(a) = \int_0^1 \frac{x^{a-1}+x^{-a}}{1+x}dx = \frac{1}{2} \int_0^\infty \frac{x^{a-1}+x^{-a}}{1+x}dx. \quad (3)\)

(ii) The function \(B(a)\) has a local minimum at \(a = 1/2\) and the minimum value is \(\pi\). The function \(B'(a)\) is strictly increasing on \((0, 1)\).

**Proof.** Break the integral (2) into two integrals, over \((0, 1)\) and \((1, \infty)\). In the latter put \(x = 1/t\) and the result follows.

To prove (ii), differentiate the identity \(B(a) = B(1-a)\) and substitute \(a = 1/2\) to get \(B'(1/2) = 0\). The second derivative is strictly positive on \((0, 1)\) so that the first derivative is strictly increasing, vanishing exactly once at \(1/2\). Finally the integral \(B(1/2)\) is elementary and its value is seen to be \(\pi\).

A rescaling of the variable of integration in (3) produces:

**Lemma 2.** In what follows \(B(a)\) would be abbreviated as \(B\).

(i) \(w^{a-1}B = \int_0^\infty \frac{x^{a-1}dx}{x+w} \quad (4)\)

(ii) \(w^{-a}B = \int_0^\infty \frac{x^{a-1}dx}{1+xw} \quad (5)\)

**Proof.** Setting \(x = t/w\) in (2) gives (i) and the substitution \(x = tw\) transforms (2) into (ii).

Equation (3) suggests adding (4) and (5), dividing through by \(1/(w+1)\) and integrating over \((0, \infty)\). The easy calculation results in:

**Lemma 3.**

$$B^2 = \int_0^\infty \frac{x^{a-1}\log xdx}{x-1} \quad (6)$$

We are now ready to obtain the ODE for \(B(a)\).

**Lemma 4.** \(B(a)\) satisfies the following initial value problem for ODEs:

$$BB'' = (B')^2 + B^4, \quad B(1/2) = \pi, \quad B'(1/2) = 0. \quad (7)$$

**Proof.** Owing to the symmetry \(B(a) = B(1-a)\) there is no loss of generality in assuming that \(0 < a < 1/2\). Subtract (5) from (4), divide by \((w-1)\) and integrate with respect to \(w\) over \((0, \infty)\) to get

$$B \int_0^\infty \frac{w^{a-1}-w^{-a}}{w-1}dw = 2 \int_0^\infty \frac{x^{a-1}\log xdx}{x+1} = 2 \frac{dB}{da}. \quad (8)$$
But from (6), the left hand side of the last equation is exactly

\[ B \int_{1-a}^{a} B^2(t) dt, \]

and (8) assumes the form

\[ B \int_{1-a}^{a} B^2(t) dt = 2 \frac{dB}{da} \]

which, in view of the symmetry \( B(t) = B(1-t) \) may be rewritten as

\[ B \int_{1/2}^{a} B^2(t) dt = \frac{dB}{da} \] (9)

Differentiating (9) with respect to \( a \) gives immediately (7).

**Theorem 10.** For \( 0 < a < 1 \), \( \Gamma(a) \Gamma(1-a) = \pi \csc \pi a \)

**Proof.** The differential equation (7) admits two elementary integrations as we now show. We work in the interval \([1/2, 1]\) and denote \( B' \) by \( C \). Since \( B' > 0 \) on \([1/2, 1]\) we may regard \( C \) as a function of \( B \). Using the chain rule,

\[ B'' = \frac{dC}{da} = \frac{dC}{dB} \frac{dB}{da} = C \frac{dC}{dB} \]

This transforms (7) into a simple linear ODE for \( C^2 \) namely,

\[ \frac{1}{2} \frac{dC^2}{dB} = \frac{C^2}{B} + B^3, \quad C(1/2) = 0. \]

Since \( C > 0 \) on \((1/2, 1)\), this gives

\[ C = \frac{dB}{da} = B \sqrt{B^2 - \pi^2} \quad (a > 1/2) \]

and the result follows upon integration.

**Acknowledgement**

The author thanks Mr. Aman Abhishek from the Department of physics, IIT Roorkee for working through the paper.

**References**


---

**Remembering K. Ramachandra**

Matti Jutila

Department of Mathematics, University of Turku, FI-20014 Turku, Finland

E-mail: jutila@utu.fi

I met Professor Ramachandra for the first time in September 1971 in Moscow, where we participated in a number theory conference in honor of the 80th birthday of academician I. M. Vinogradov. By the way, we met again in 1981 at a similar occasion celebrating the 90th birthday of Vinogradov. We had common interests in number theory, in particular prime numbers and Dirichlet series, so our scientific contacts and personal friendship got started immediately and lasted forty years until Ramachandra’s death. His talk [7] in Moscow turned out to be quite influential for me, and therefore I would like to dwell for a moment on this topic and related questions. Ramachandra considered the following problem: given \( k \) consecutive numbers \( n+1, \ldots, n+k \), what is the biggest prime factor, say \( P \), occurring in these numbers? Erdős had proved in an elementary way that \( P \gg k \log k \), but improving this bound turned out to be a highly nontrivial problem, as commented by Erdős himself. As a reformulation, one may ask about gaps between numbers with a large prime factor. Extreme cases of such numbers are the prime numbers \( p_n \) themselves, and by the classical theory of Ingham we have a relation between zero-density estimates for Riemann’s zeta-function \( \zeta(s) \) and the gaps between primes. Let \( N(\alpha, T) \) denote the number of zeros of \( \zeta(s) \) in the domain \( \text{Re } s \geq \alpha, |\text{Im } s| \leq T \). The density hypothesis

\[ N(\alpha, T) \ll T^{2(1-\alpha)+\varepsilon} \]

for \( 1/2 \leq \alpha \leq 1 \) and \( T \geq 1 \) implies the bound (see [1], eq. (12.83))
\( p_{n+1} - p_n \ll p_n^{1/2+\varepsilon}. \)  \hspace{1cm} (2)

However, the density hypothesis is still unproved; I proved it for \( \alpha \geq 11/14, \) and this bound has been improved somewhat by J. Bourgain. Now, as an enlargement of the sequence of primes, consider the sequence of \( r_n^{(\beta)} \) of the numbers \( r \) having a prime factor \( p \geq r^{\beta} \) for given \( \beta \in (0,1). \) As an analogue of (2), one may now pose the following problem: to find \( \beta, \) as large as possible, such that (2) holds if \( p_n \) is replaced by \( r_n^{(\beta)}. \) Ramachandra had shown this for \( \beta = 5/8, \) and in [3]. I obtained (2) for \( \beta = 2/3. \) My argument was based on a weighted density theorem, that is a density theorem where the zeros \( \rho \) are weighted by \( |f(\rho)| \) with \( f(s) \) a given Dirichlet polynomial. This approach fails to give any result for gaps of length not exceeding the square root of the numbers in question. On the other hand, Ramachandra’s method, which was based on Selberg’s sieve and van der Corput’s method, was applicable more generally. The same was true for my paper [4], where I used Vinogradov’s method for exponential sums over primes. However, for extremely large \( n \) compared with \( k, \) another powerful tool was needed, namely Baker’s method from the transcendental number theory, and this very delicate case was treated by Ramachandra and Shorey. As a combination of all cases, the final result was

\[ P \gg k \log k \log \log k / \log \log \log k, \]

a sharpening of the above mentioned result of Erdős. Interestingly, its proof required combining methods of essentially different nature. Another situation like this will be discussed next, namely estimating gaps between primes.

Let \( \theta \) be a number such that for any positive \( \varepsilon \) we have

\[ p_{n+1} - p_n \ll p_n^{\theta+\varepsilon}. \]

The value \( \theta = 7/12 \) (see [1], eq. (12.68)) due to M. N. Huxley is presently the best result obtained by complex analytic methods using zero-density estimates for the zeta-function. Henryk Iwanie and myself were visiting the Institute Mittag-Leffler in Stockholm in 1977-8, and Iwaniec made an attempt to apply his version of the linear sieve to gaps between primes. It turned out that a certain version of the weighted density theorem was helpful in this context, and in [2] we showed that \( \theta = 13/23 \) is admissible. Note that \( 7/12 = 0.5833 \ldots, \) while \( 13/23 = 0.5652 \ldots; \) the last mentioned value has been improved afterwards by more refined techniques. It is fair to say that Ramachandra was indirectly involved in this development, for his problem on numbers with a large prime factor motivated me to consider weighted density theorems, and such a device found unexpectedly an application to the gap problem for primes.

One of Ramachandra’s favorite topics was estimating moments of the zeta-function. A typical case is

\[ I_k(T) = \int_{1-\varepsilon}^T |\zeta(1/2 + it)|^{2k} dt, \]

and one may also consider moments over short intervals and over different lines. The cases \( k = 1,2 \) are classical and asymptotic formulae for \( I_k(T) \) are known, whereas the other cases are more problematic. D. R. Heath-Brown proved that on Riemann’s Hypothesis

\[ T(\log T)^{k^2} \ll I_k(T) \ll T(\log T)^{k^2} \space \hspace{1cm} (3) \]

for \( 0 \leq k \leq 2. \) Moreover, the lower bound holds for all positive \( k. \) The implied constants here depend on \( k. \) As to unconditional results, the lower bound holds if \( k \) is rational, and the same is true for the upper bound if \( k = 1/n, \) where \( n \) is a natural number. My contribution [5] to the topic was that if \( k = 1/n, \) then the implied constants in (3) can be taken to be absolute, thus independent of \( n. \) As the choice of \( n \) is now flexible, one may deduce results on the value distribution of the zeta-function. Again the work of Ramachandra was of pioneering importance, and his monograph [8] gives an account of moment problems together with other related topics.

In the context of the fourth moment of the zeta-function, Ramachandra’s reflection principle should be mentioned (see [1], Sec. 4.4). It is a flexible variant of the approximate functional equation for \( \zeta^2(s), \) and since its proof requires only the functional equation, it can be immediately generalized to other Dirichlet series having a functional equation of the Riemann type. This device is now a standard tool in analytic number theory.

In 1985, Ramachandra kindly arranged for me a position as a visiting professor at the Tata Institute of Fundamental Research, a fortunate opportunity indeed to work at this highly reputed scientific center. That time I was...
developing a method, combining ideas of the methods of Hardy-Littlewood and van der Corput, in the theory of exponential sums involving the divisor function or Fourier coefficients of cusp forms. My lectures [6] on this topic appeared in the Tata Lecture Notes series, and I felt it as a great honor to have a book published in a series including authors such as C. L. Siegel and many other famous names. Curiously, Bombieri and Iwaniec were working at the same time on an analogous method for zeta-sums, that is for segments of the zeta-series.

Ramachandra organized visits to the Tata Institute for several other number theorists as well, so my friends Y. Motohashi, A. Ivic and M. N. Huxley were enjoying his hospitality. Moreover, Motohashi and Ivic also published their lectures in the prestigious Tata series.

Professor Ramachandra carried on his mathematical activity with full force even after retirement. Indeed, last year I was still in contact with him when he informed me that my paper on the estimation of the Mellin transform of Hardy’s function had been accepted for publication in the Hardy–Ramanujan Journal. Therefore the sad message that he had passed away came so unexpectedly for me, and my sorrow was mixed with the relief that he was fortunate to retain and continue his devotion to mathematics till the very end of his life.

References


Ramachandra: Personal Reminiscences (1933–2011)

K. Srinivas

Institute of Mathematical Sciences, CIT Campus, Tharamani, Chennai 600 113, India
E-mail: srini@imsc.res.in

There are some who do mathematics and there are others who live in mathematics. Ramachandra belonged to the second category. Mathematics was everything for him. The analytical properties of the Riemann zeta-function, \( \zeta(s) \), was something that was close to his heart. The ease with which he would derive deep properties of \( \zeta(s) \), many times using only simple arguments like maximum modulus principle, Cauchy’s residue theorem, Hölder’s inequality, partial summation formula … exhibits his deep understanding of the subject. One of the missions he undertook in recent years was to establish the known properties \( \zeta(s) \) without using functional equation for \( \zeta(s) \)! To strengthen his point he established the proof of the infinitude of simple zeros of \( \zeta(s) \) on the critical line in a very simple, yet elegant way without the use of functional equation. Another desire he had was to see a simple proof of Fermat’s Last Theorem.

He believed in the classical Indian Gurukul system at heart. His Gurus were Srinivasa Ramanujan, G. H. Hardy, Paul Erdős, I. M. Vinogradov, Carl L. Siegel, Atle Selberg, Roger Heath-Brown, … just to name a few. The name of Srinivasa Ramanujan, in particular, would overwhelm his heart with emotions and sometimes tears would roll out from his eyes.
On one occasion, he was invited to deliver a talk on the works of Ramanujan in the Homi Bhabha auditorium, T.I.F.R. He was speechless for considerable amount of time as he was displaying the picture of Srinivasa Ramanujan. He was greatly influenced and affected by the life and works of Ramanujan. He tried to imbibe the values derived from Ramanujan’s life. Like the famous taxi-cab number 1729, Ramachandra discovered, while in college, that \(3435 = 3^3 + 4^4 + 3^3 + 5^5\) is the only number \(> 1\) with then property†

\[
x_n 10^n + x_{n-1} 10^{n-1} + \cdots + 10x_1 + x_0 = x_n^{x_n} + x_{n-1}^{x_{n-1}} + \cdots + x_0^{x_0}.
\]

†Here \(0^0 = 1\), if \(0^0 = 0\) then 438579088 is the only other number with the above property (see http://en.wikipedia.org/wiki/Perfect_digit-to-digit_invariant)

Apparently, his college principal had a car with the number 3430 displayed in the number plate and by adding 5 to it Ramachandra figured out this property! To express his gratitude towards Ramanujan and also to facilitate publication of quality papers purely devoted to analytic number theory, he (along with R. Balasubramanian) founded the Hardy-Ramanujan society and brought out the Hardy Ramanujan journal starting from 1978. This journal attracts a number of good papers from reputed mathematicians. A section of this journal is devoted to popularize some important results which deserve to be well known. As a token of appreciation for good work, they instituted the Distinguished Award of the Hardy Ramanujan society. Over the years the award has achieved considerable respect in the mathematical community.

It is said that without him Analytic number theory would have disappeared from the Indian scene in the seventies. His brave fight to defend the subject and the interests of fellow number theorists eventually resulted in the formation of a strong analytic number theory group in India which enjoys considerable international reputation. He was greatly concerned about the future of analytic number theory in India. It was his dream that in the land of Ramanujan, analytic number theory should flourish beyond limits. To popularize analytic number theory he wrote several expository articles on topics like Riemann Hypothesis, Brun sieve, prime number theorem, Waring’s problem, Ramanujan’s circle method . . . . I hope that the analytic number theory group would take his mantle forward in the years to come and make his dreams come true.

To his credit, he has about two hundred published research papers and four books on number theory. His PhD
students include S. Srinivasan, T. N. Shorey, M. Narlikar, R. Balasubramanian, V. V. Rane, A. Sankaranarayanan, K. Srinivas and Kishor Bhat. I was fortunate to have him as my thesis advisor. As an advisor, he was completely different from others. He would propose a problem and would enquire about the progress at least three times a day, everyday, until one day he would himself declare that this is enough for a paper! Before we would get into a celebration mood, which invariably meant to have a coffee and share a cake in the west canteen of T.I.F.R, he would be ready with either an extension of the current problem or posing a different problem and this routine continued until I finished my thesis. This aspect of his personality shows his ability to initiate certain themes which paved way for others to pursue further and thereby enriching the literature. He would advise in a fatherly way to go deep into a particular topic, rather than accumulating a superficial knowledge of too many things.

The most remarkable quality of him was his simplicity, honesty and compassion for fellow human beings. I have never heard him boasting about himself or his achievements. He would, in fact, speak very fondly about the achievements of his students. Even though he commanded great respect from all, he never succumbed to the temptations of power and greed. The demise of Ramachandra is not only an irreparable loss for his family but also to the entire mathematical community.

On Harmonic and Quasiregular Mappings

S. Ponnusamy
Department of Mathematics, Indian Institute of Technology Madras, Chennai 600 036, India
E-mail: samy@iitm.ac.in

Antti Rasila
Antti Rasila, Institute of Mathematics, Aalto University, P. O. Box. 11100, FI-00076 Aalto, Finland
E-mail: antti.rasila@iki.fi

Abstract. The main purpose of this article is to continue the discussion on harmonic mappings from our three earlier articles [22–24]. Our previous articles have mainly dealt with the aspects of the theory of harmonic mappings which arise from the classical complex analysis. But in order to have a sound understanding of harmonic mappings, one also needs some understanding of the theory of quasiconformal mappings, as many important concepts and methodologies on harmonic mappings arise from this context. In addition, we present some ideas on hyperbolic related metrics whose presence in connection with planar harmonic mappings are not considered.

Keywords. Harmonic, analytic, univalent and convex functions

2000 MSC. Primary: 30C62; Secondary: 30C45, 30C85, 31A05

1. Quasiconformal and quasiregular mappings

In this section, we present some of the basic definitions and properties of quasiconformal and quasiregular mappings in the complex plane. Quasiconformal and quasiregular mappings are, in a geometric sense, a natural generalization of conformal mappings and analytic functions, respectively. There are certain differences between the theories of harmonic mappings and quasiconformal mappings, for example the composition of quasiconformal mappings is always quasiconformal whereas the same is not true in the case of harmonic mappings. On the other hand, these classes of mappings do not have some of the nice properties of the harmonic mappings, such as series representations. The class of harmonic quasiconformal mappings was first introduced by Martio [19].
Quasiconformal diffeomorphisms were first introduced by H. Grötzsch as a technical tool for studying a problem concerning conformal mappings, but this class of mappings has independent interest. In fact, many of the results on conformal mappings, only require quasiconformality and hence, it is of interest to know when conformality is necessary and when it is not.

1.1 Grötzsch problem and quasiconformal diffeomorphisms

First we recall the problem of Grötzsch:

**Problem.** Let $Q$ be a square and $R$ be a rectangle which is not a square. In 1928, H. Grötzsch, posed the following problem: *Does there exist a conformal mapping of $Q$ onto $R$ which maps vertices onto vertices?*

By the Carathéodory extension theorem a conformal mapping between two Jordan domains has always a homeomorphic boundary extension. While conformal mapping of $Q$ onto $R$ exists by the Riemann mapping theorem, it does not guarantee a solution to this problem because of the extra condition that the vertices mapping onto vertices. Thus, the problem of Grötzsch is interesting in itself. Actually, there does not exist a conformal mapping of $Q$ onto $R$ taking vertices onto vertices. This extra condition led to the development of the theory of quasiconformal mappings.

Then the question is to find most nearly conformal mapping of this kind, and this needs a measure of approximate conformality. Let $f : \Omega \to f(\Omega)$ be a sense-preserving $C^1$-diffeomorphism, $z_0 \in \Omega$, and $w = f(z) = u(z) + i v(z)$. Then,

$$ dw = df = f_z dz + f_{\bar{z}} d\bar{z} $$

and, since $f$ is a diffeomorphism, it is locally linear (in this case at $z_0$). Indeed, the affine map $L$ defined by

$$ L(z) := f_z(z_0)dz + f_{\bar{z}}(z_0)d\bar{z} $$

sends a circle with center 0 in the $dz$-plane onto an ellipse in the $dw$-plane, with major axis of length $L = |f_z(z_0)| + |f_{\bar{z}}(z_0)|$ and minor axis of length $l = |f_z(z_0)| - |f_{\bar{z}}(z_0)|$. Note that, because $f$ is sense-preserving, the Jacobian $J_f = |f_z|^2 - |f_{\bar{z}}|^2$ is positive. It follows that

$$ (|f_z| - |f_{\bar{z}}|)|dz| \leq |dw| \leq (|f_z| + |f_{\bar{z}}|)|dz|, $$

where both the limits are attained. The differential $dw$ maps the circle $|dz| = r$ onto the ellipse, see Figure 1. The ratio $D_f(z)$ between the major and the minor axes is

$$ D_f := \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|} \geq 1. $$

The quantity $D_f(z)$ is called *dilatation* of $f$ at the point $z \in \Omega$.

We shall use the dilatation $D_f(z)$ as the measure of the approximate conformality of the mapping. Note that $f$ is a conformal mapping if and only if $D_f(z) = 1$ for all $z \in \Omega$.

Now we are ready to give a solution to a more general version of the Grötzsch problem.

Suppose $R, R'$ be two rectangles with sides $a, b$ and $a', b'$. Suppose that $f \in C^1(R)$ (i.e. continuously differentiable on $R$) takes the $a$-sides of $R$ onto the $a'$-sides of $R'$ and the $b$-sides onto the $b'$-sides (see Figure 2).

We may assume that

$$ \frac{a}{b} \leq \frac{a'}{b'}.$$  

Fix $y \in (0, b)$ and consider a curve $\gamma(x) = x + iy, x \in [0, a]$. Write $\gamma' = f \circ \gamma$. Then

$$ a' \leq |f(a + iy) - f(iy)| \leq \int_0^a df(\gamma(x)) dx $$
Definition 1. A sense-preserving diffeomorphism \( f \) is said to be \( K \)-quasiconformal on \( \Omega \) if \( D_f(z) \leq K \) throughout the given region \( \Omega \), where \( K \in [1, \infty) \) is a constant.

Example 1. The affine mapping

\[
 f(z) = \alpha z + \beta \overline{z}, \quad |\alpha| > |\beta|,
\]

is \( K \)-quasiconformal with

\[
 K = D_f(z) = \frac{|\alpha| + |\beta|}{|\alpha| - |\beta|}.
\]

This mapping is not conformal if \( \beta \neq 0 \). In particular, if we choose

\[
 \alpha = \frac{1}{2} \left( \frac{a'}{a} + \frac{b'}{b} \right), \quad \beta = \frac{1}{2} \left( \frac{a'}{a} - \frac{b'}{b} \right),
\]

we obtain a \( K \)-quasiconformal mapping of \( R \) onto \( R' \), where

\[
 K = \frac{a'/b'}{a/b}
\]

is the best possible dilatation for a solution of the Grötzsch problem.

1.2 Definitions of quasiconformality

While harmonic quasiregular mappings are always diffeomorphisms, it should be noted that the class of quasiconformal diffeomorphisms is not the natural class to study when considering quasiconformal mappings independently from the theory of harmonic mappings. For example, it is easy to construct a sequence of quasiconformal homeomorphisms converging to, e.g., a piecewise linear mapping, which does not have partial derivatives at certain points. Next we will give some standard definitions of quasiconformality. These definitions are required to study the general class of quasiconformal mappings. We start with the so-called metric definition, which is perhaps the most intuitive way to understand quasiconformality.

Let \( \Omega_1, \Omega_2 \) be domains in the extended complex plane \( C_\infty = \mathbb{C} \cup \{\infty\} \), and suppose that \( f : \Omega_1 \to \Omega_2 \) is a sense-preserving homeomorphism. For each \( z \in \Omega_1 \backslash \{\infty, f^{-1}(\infty)\} \), we define linear dilatation of \( f \) at \( z \) by

\[
 H_f(z) = \limsup_{r \to 0} \frac{L_f(z, r)}{\ell_f(z, r)},
\]

where

\[
 L_f(z, r) = \max_{|z-w|=r} |f(z) - f(w)|,
\]

and

\[
 \ell_f(z, r) = \min_{|z-w|=r} |f(z) - f(w)|.
\]

Definition 2. We say that \( f \) is \( K \)-quasiconformal, if \( H_f \) is uniformly bounded in \( \Omega_1 \backslash \{\infty, f^{-1}(\infty)\} \), i.e.

\[
 H_f(z) \leq K,
\]

where \( 1 \leq K < \infty \) is a constant (not depending on \( z \)).

It is possible to generalize the definition to the case where the mapping is defined in \( \mathbb{R}^n \) or even in a general metric space. In \( \mathbb{R}^n \), by a result of Heinonen and Koskela [16], \( \lim sup \) can be replaced with \( \lim \inf \). For an expository article on the theory of
quasiregular maps in the $n$-dimensional Euclidean space $\mathbb{R}^n$, we refer to paper of Väisälä [27]. The article also contains a collection of open problems and a representative list of literature on quasiregular maps.

**Remark 1.** A homeomorphism $f : \Omega \to f(\Omega)$ satisfying

$$|z - w|/L \leq |f(z) - f(w)| \leq L|z - w|$$ for all $z, w \in \Omega$

is called $L$-bilipschitz. It is easy to see that $L$-bilipschitz mappings are $L^2$-quasiconformal.

Not all quasiconformal mappings are bilipschitz. The standard counterexample is the radial stretching:

$$f(z) = |z|^\alpha z, \quad f(0) = 0, \quad \alpha \in (0, 1),$$

which is $K$-quasiconformal with $\alpha = 1/K$.

Next we will give a very useful analytic characterization of quasiconformality. For example, from this definition it is easy to see when a harmonic mapping is quasiconformal. We do not require for a quasiconformal mapping to be differentiable to define the class $ACL(\Omega)$ of functions on $\Omega \subset \mathbb{C}$ which are absolutely continuous on lines. By absolute continuity we mean the following.

**Definition 3.** Let $f$ be a complex valued function defined on subinterval $[a, b]$ of $\mathbb{R}$. Suppose that for a given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sum_{k=1}^{n} |f(a_k) - f(b_k)| < \varepsilon$$

for all $a = a_1 < b_1 \leq \cdots < a_n < b_n = b$ such that

$$\sum_{k=1}^{n} |a_k - b_k| < \delta.$$

Then $f$ is said to be absolutely continuous in $[a, b]$. Obviously, an absolutely continuous function is continuous.

The class $ACL(\Omega)$ is defined as follows. The definition is somewhat technical, the motivation is to guarantee the existence of weak partial derivatives.

**Definition 4.** Let $\Omega$ be a domain in $\mathbb{C}$. We say that a function $f : \Omega \to \mathbb{R}$ is absolutely continuous on lines ($ACL$) in $\Omega$ if, for each rectangle

$$R = [a, b] \times [c, d] \subset \Omega,$$

the function $f(x + iy)$ is absolutely continuous with respect to the variable $y$ for a.e. $x \in [a, b]$ and with respect to $x$ for a.e. $y \in [c, d]$. A complex valued function is absolutely continuous on lines if its real and imaginary parts are $ACL$.

Now we are ready to state the analytic definition of quasiconformality.

**Definition 5.** A sense preserving homeomorphism $f : \Omega_1 \to \Omega_2$ is $K$-quasiconformal if and only if $f$ is in $ACL(\Omega_1)$ and

$$\max_u |\partial_uf(z)|^2 \leq K J_f(z) \quad \text{a.e. in } \Omega_1,$$

where $\partial_uf = f_z + f_{\bar{z}} e^{-2iu}$ denotes the directional derivative of $f$ in the direction $\alpha$ and $J_f(z)$ is the Jacobian of $f$ at $z$.

If the inequality (1) holds for an $ACL(\Omega_1)$ mapping $f$ a.e. in $\Omega_1$, but $f$ is not necessarily homeomorphic, then $f$ is called $K$-quasiregular.

A mapping is said to be quasiconformal if it is $K$-quasiconformal for some $K \geq 1$. It is easy to see that an analytic function is always 1-quasiregular, and 1-quasiconformal if it is univalent.

**Remark 2.** By a result of S. Stoilow a quasiregular mapping $f$ of $\mathbb{D}$ onto a domain $\Omega$ can be represented as $f = g \circ h$, where $h$ is a quasiconformal mapping of $\mathbb{D}$ onto itself and $g$ is an analytic function (see e.g. [17] or [2]).

For this reason the quasiregular mappings of the plane have little independent interest. The situation is very different in $\mathbb{R}^n$ for $n \geq 3$. In fact, by generalized Liouville’s theorem, for $n \geq 3$ every 1-quasiregular mapping is a restriction of a Möbius transformation, or a constant.

The ratio $\mu_f = f_z/f_{\bar{z}}$ is called the complex dilatation of $f$. If $f$ is sense-preserving, then $|\mu_f(z)| \in [0, 1)$. It may be observed that a sense-preserving diffeomorphism $f : \Omega \to \mathbb{C}$, e.g., a sense-preserving harmonic mapping, is $K$-quasiconformal if and only if

$$|\mu_f(z)| \leq \frac{K - 1}{K + 1}.$$

It follows that $f$ is quasiconformal if and only if $|\mu_f(z)| \leq k < 1$. The mapping $f$ is conformal if and only if $\mu_f = 0$.

One approach to the theory of planar quasiconformal mappings based on the following fundamental result which is called the measurable Riemann mapping theorem.
Theorem 1 (see e.g. [2] or [17]). Suppose that \( \mu : \mathbb{C} \to \mathbb{C} \) is a measurable function with \( |\mu(z)| \leq k < 1 \) for a.e. \( z \in \mathbb{C} \). Then there exists a solution \( f : \mathbb{C}_\infty \to \mathbb{C}_\infty \) to the Beltrami equation

\[
f_z(z) = \mu(z) f_z(z) \quad \text{for a.e. } z \in \mathbb{C}.
\]

The solution is a \( K \)-quasiconformal homeomorphism, normalized by the three conditions \( f(0) = 0 \), \( f(1) = 1 \) and \( f(\infty) = \infty \). Moreover, the normalized solution is unique up to an additive constant.

Various versions of this result exist in the literature. In particular, there are versions involving the so-called degenerate Beltrami equation, i.e., it is only required that \( |\mu(z)| < 1 \). For a comprehensive survey of this topic see [2].

We end this section with the following basic result concerning planar harmonic mappings.

**Theorem 2.** Let \( f \in C^2(\Omega) \) with \( J_f(z) > 0 \) on \( \Omega \), where \( \Omega \) is a simply connected domain \( \Omega \subset \mathbb{C} \). Then the function \( f \) is harmonic on \( \Omega \) if and only if \( f \) is the solution of the elliptic partial differential equation

\[
\frac{\partial f}{\partial z} = \omega(z) \frac{\partial f}{\partial \overline{z}},
\]

for some analytic function \( \omega \) on \( \Omega \) with \( |\omega(z)| < 1 \) on \( \Omega \).

We point out that every harmonic function \( f : \Omega \to \mathbb{C} \) in a simply connected domain \( \Omega \subset \mathbb{C} \) admits the representation

\[
f = h + \overline{g},
\]

where \( h \) and \( g \) are analytic in \( \Omega \). The representation is unique up to an additive constant.

2. Conformal modulus and capacity

Suppose that \( \Gamma \) is a path family in \( \mathbb{C}_\infty \). We will assign a conformally invariant quantity, called modulus, to \( \Gamma \), which measures the conformal size of \( \Gamma \).

**Definition 6.** Let \( \rho : \mathbb{C} \to [0, \infty) \) be a Borel measurable function. We call \( \rho \) admissible for \( \Gamma \) (denoted by \( \rho \in \mathcal{F}(\Gamma) \)) if

\[
\int_\Gamma \rho(z) |dz| \geq 1,
\]

for each locally rectifiable \( \gamma \in \Gamma \). The (conformal) modulus of \( \Gamma \) is then defined by

\[
M(\Gamma) = \inf_{\rho} \int_{\mathbb{C}} \rho(z)^2 \, dx \, dy,
\]

where the infimum is taken over all \( \rho \in \mathcal{F}(\Gamma) \).

2.1 Properties of the conformal modulus

We will begin with a few basic properties of the conformal modulus. The conformal modulus is very useful in studying quasiconformal mappings.

**Lemma 3.** The modulus is an outer measure in the space of all path families in \( \mathbb{C} \), i.e.

1. \( M(\emptyset) = 0 \),
2. If \( \Gamma_1 \subset \Gamma_2 \) then \( M(\Gamma_1) \leq M(\Gamma_2) \), and
3. \( M(\bigcup_j \Gamma_j) \leq \sum_j M(\Gamma_j) \).

We say that \( \Gamma_2 \) is minorized by \( \Gamma_1 \) and write \( \Gamma_1 \prec \Gamma_2 \) if every \( \gamma \in \Gamma_2 \) has a subpath in \( \Gamma_1 \).

**Lemma 4.** If \( \Gamma_1 \prec \Gamma_2 \) then \( M(\Gamma_1) \geq M(\Gamma_2) \).

**Proof.** If \( \Gamma_1 \prec \Gamma_2 \) then obviously \( \mathcal{F}(\Gamma_1) \subset \mathcal{F}(\Gamma_2) \). Hence

\[
M(\Gamma_1) \geq M(\Gamma_2).
\]

**Lemma 5.** If \( \Gamma \) is a family of paths in \( \Omega \) such that \( \ell(\gamma) \geq r \), then \( M(\Gamma) \leq m(\Omega) r^{-2} \), where \( m(\Omega) \) is the two-dimensional Lebesgue measure on \( \Omega \).

**Proof.** Follows immediately from (2) and the fact that the function \( \rho = \chi_\Omega / r \), where

\[
\chi_\Omega(z) = \begin{cases} 
1 & \text{for } z \in \Omega, \\
0 & \text{otherwise},
\end{cases}
\]

is admissible for \( \Gamma \).
The inverse mapping is very useful in the study of conformality by using the conformal modulus. This definition is also made in a geometric way. Let us consider the function \( f : \Omega_1 \rightarrow \Omega_2 \). If \( f \) is conformal, for every family of paths \( \gamma \in \Gamma \),

\[
\int_{\gamma} \rho \, |dz| = \int_{\gamma} \rho_1 (f(z)) |f'(z)| \, |dz| = \int_{f \circ \gamma} \rho_1 \, |dz| \geq 1.
\]

Hence, \( \rho \in \mathcal{F}(\Gamma) \), and

\[
M(\Gamma) \leq \iint_{\Gamma} \rho^2 \, dx \, dy = \sum_{i=1}^{n} \int_{\Omega_i} \rho_i^2 (f(z)) |J_f(z)| \, dx \, dy = \iint_{\Omega_2} \rho_2^2 \, dx \, dy = \sum_{i=1}^{n} \int_{\Omega_i} \rho_1^2 \, dx \, dy
\]

for all \( \rho_1 \in \mathcal{F}(f(\Gamma)) \), and thus \( \mathcal{M}(\Gamma) \leq \mathcal{M}(f(\Gamma)) \).

The inverse inequality follows from the fact that \( f^{-1} \) is conformal.

### 2.3 Geometric definition of quasiconformality

Now we are ready to give a characterization of quasiconformality by using the conformal modulus. This definition is also very useful in the geometric theory of quasiconformal mappings. In particular, this technique is used for obtaining lower bounds for the modulus in many situations. Estimates of this type can be used in the geometric theory of quasiconformal mappings. In particular, this technique is used for obtaining distortion results for quasiconformal mappings. First we introduce the concept of conformal the modulus of a so-called quadrilateral.

#### Definition 7.

A Jordan domain \( \Omega \) in \( \mathbb{C} \) with marked (positively ordered) points \( z_1, z_2, z_3, z_4 \) \( \in \partial \Omega \) is a quadrilateral and denoted by \( (\Omega; z_1, z_2, z_3, z_4) \). We use the (unique) conformal mapping of quadrilateral onto a rectangle \( (\Omega'; 1 + ih, ih, 0, 1) \), with the vertices corresponding, to define the modulus \( \mathcal{M}(Q) = h \) of \( Q = (\Omega; z_1, z_2, z_3, z_4) \).

Note that, by the Carathéodory extension, a conformal mapping of a Jordan domain \( \Omega \) onto a rectangle has a homeomorphic boundary extension, so the images of the boundary points are well defined. Obviously, the modulus of \( (\Omega; z_2, z_3, z_4, z_1) \) is \( 1/h \).

Moduli of quadrilaterals and path families are connected to each other as follows.

#### Lemma 8.

Let \( Q = (\Omega; z_1, z_2, z_3, z_4) \) be a quadrilateral and denote by \( \gamma_j \) the boundary arc of \( \Omega \) connecting \( z_j \) and \( z_{j+1} \).
for \( j = 1, 2, 3 \) and \( z_4, z_0 \) for \( j = 4 \). Then \( M(Q) = 1/M(\Gamma) \), where \( \Gamma \) is the family of paths connecting \( \gamma_2 \) and \( \gamma_4 \) in \( \Omega \).

### 2.5 Ring domains

A domain \( \Omega \) in \( \mathbb{C} \) (or \( \mathbb{C}_\infty \)) is called a ring domain if it has exactly two boundary components. If the boundary components are \( E \) and \( F \), we denote the ring domain by \( R(E,F) \). Then by the Riemann mapping theorem for doubly connected domains, every ring domain \( \Omega \) can be mapped conformally onto the canonical annulus

\[ A_r = \{ z : 1 < |z| < r \}, \]

where \( r \in (0, \infty) \). Then the number \( \text{mod}(\Omega) = \log r \), that determines the conformal equivalence class of \( \Omega \), is called the conformal modulus of the ring domain \( \Omega \). See also [17, I.6].

For \( E, F, \Omega \subset \mathbb{C}_\infty \), we denote by \( \Delta(E,F;\Omega) \) the family of all nonconstant paths joining \( E \) and \( F \) in \( \Omega \). The conformal modulus of a ring domain \( R = R(E,F) \) is defined by

\[ \text{mod}(R) = \frac{2\pi}{M(\Delta(E,F;\Omega))}, \]

provided that \( M(\Delta(E,F;\Omega)) \neq 0 \). Otherwise, we define \( \text{mod}(R) = \infty \). Note that this happens only if either \( E \) or \( F \) is a singleton.

**Example 2.** Let \( 0 < a < b < \infty \), and let \( A = \mathbb{D}(b)\setminus\mathbb{D}(a) \). Then

\[ \text{mod}(R) = \frac{2\pi}{M(\Gamma_A)} = \log \frac{b}{a}, \]

where \( \Gamma_A = \Delta(S(a), S(b); A) \).

Every ring domain \( R \) can be mapped conformally onto the annulus \( \{ z : 1 < |z| < e^M \} \), where \( M = \text{mod}(R) \) is the conformal modulus of the ring domain \( R \). If we map the annulus

\[ A_r = \{ z : 1 < |z| < r \}, \]

with the segment \( [1, r] \) on the real axis removed, by

\[ z \mapsto \log z = \log |z| + i \arg z, \quad 0 < \arg z < 2\pi, \]

the image is the rectangle \( R \) with vertices \( (\log r + 2\pi i, 2\pi i, 0, \log r) \). We see that

\[ M(R; \log r + 2\pi i, 2\pi i, 0, \log r) = \frac{2\pi}{\log r}. \]

Thus \( 2\pi / \log r \) has an interpretation as the modulus of a quadrilateral (see also [17, p. 36]).

**Remark 3.** Let \( D \) be a quadrilateral such that \( \partial D \subset \Omega \). Then, for a \( K \)-quasiconformal mapping \( f \) of a domain \( \Omega \) onto \( f(\Omega) \),

\[ \frac{M(D)}{K} \leq M(f(D)) \leq KM(D). \]

In fact, a sense-preserving homeomorphism \( f : \Omega_1 \to \Omega_2 \) is quasiconformal if and only if the modules of quadrilaterals are \( K \)-quasi-invariant in the above sense. The result remains true if we replace the quadrilateral \( D \) by a ring domain.

### 2.6 Numerical methods

The computation of the moduli of quadrilaterals and ring domains with piecewise smooth boundaries is a problem frequently occurring in various applications, see [21]. There is no general method for such computations except perhaps the case of polygonal quadrilaterals when the canonical conformal mapping can be found by using the SC Toolbox [9,10] may be considered as the “state-of-the-art” tool. Other methods for numerical computation of conformal mappings are discussed in the chapter of Nicolas Papamichael [21].

Conformal modulus of a quadrilateral can also be solved by studying the following Dirichlet-Neumann (mixed boundary value) problem. Suppose that \( \Omega \) is a domain such that \( \partial \Omega \) consists of a finite number of regular Jordan curves of the boundary a normal \( \partial n \) is defined. Let \( \psi \) be a real-valued continuous function defined on \( \partial \Omega \), and let \( \partial \Omega = A \cup B \), where \( A, B \) both are unions of Jordan arcs. The problem is to find a function \( u \) satisfying the following conditions:

1. The function \( u \) is continuous and differentiable in \( \hat{\Omega} \).
2. \( u(t) = \psi(t), \quad t \in A \).
3. If \( \partial u / \partial n \) denotes differentiation in the direction of the exterior normal, then

\[ \frac{\partial}{\partial n} u(t) = \psi(t), \quad t \in B. \]

It is possible to express the modulus of a quadrilateral \( (\Omega; z_1, z_2, z_3, z_4) \) in terms of the solution of the Dirichlet-Neumann problem. Let \( \gamma_j \), where \( j = 1, 2, 3, 4 \), be the arcs of \( \partial \Omega \) between \( \{ z_1, z_2 \}, \{ z_2, z_3 \}, \{ z_3, z_4 \}, \{ z_4, z_1 \} \), respectively. If \( u \) is the (unique) harmonic solution of the Dirichlet-Neumann problem with boundary values of \( u \) equal to 0 on \( \gamma_2 \), equal to 1 on \( \gamma_4 \) and with \( \partial u / \partial n = 0 \) on \( \gamma_1 \cup \gamma_3 \), then

\[ M(\Omega; z_1, z_2, z_3, z_4) = \int_{\Omega} |\nabla u|^2 \, dx \, dy. \]
Similarly, for a ring domain $R$, the capacity of $R$ can be defined by

$$\text{cap } R = \inf_u \int_R |\nabla u|^2 \, dx \, dy,$$

where the infimum is taken over all nonnegative, piecewise differentiable functions $u$ with compact support in $R \cup E$ such that $u = 1$ on $E$. It is well-known that the harmonic function on $R$ with boundary values 1 on $E$ and 0 on $F$ is the unique function that minimizes the above integral. In other words, the minimizer may be found by solving the Dirichlet problem for the Laplace equation in $R$ with boundary values 1 on the bounded boundary component $E$ and 0 on the other boundary component $F$. A ring domain $R$ can be mapped conformally onto the annulus $\{z : e^{-M} < |z| < 1\}$, where $M = \text{mod}(R)$ is the conformal modulus of the ring domain $R$. The modulus and capacity of a ring domain are connected by the simple identity $\text{mod}(R) = 2\pi / \text{cap } R$.

The solutions of the Dirichlet and Dirichlet-Neumann problems can be approximated, e.g. by the method of finite elements, which allows us to obtain numerical values of the modulus. A software package which can be used for this purpose is the $h$-adaptive software package AFEM of K. Samuelsson [5]. Other experimental results obtained by using AFEM have been reported in [25,26]. Very recently, a new $hp$-FEM algorithm for the computation of moduli of rings and quadrilaterals was studied in [15]. Its accuracy and performance seem to compare well with previously known methods such as the Schwarz-Christoffel Toolbox of Driscoll and Trefethen [10] (see also [9]). The $hp$-FEM algorithm also applies to the case of non-polygonal boundary.

### 2.7 Canonical ring domains

Next we consider the so-called canonical ring domains which are very useful in studying quasiconformal mappings. For example, a version of the Schwarz lemma for quasiconformal mappings can be derived by using the properties of these ring domains.

The complementary components of the Grötzsch ring $R_G(s)$ in $\mathbb{C}_\infty$ are the unit circle $S = \{z : |z| = 1\}$ and the infinite line segment $[s, \infty)$, $s > 1$, and those of the Teichmüller ring $R_T(s)$ are $[-1, 0]$ and $[s, \infty)$, $s > 0$.

We define two special functions $\gamma(s)$, $s > 1$ and $\tau(s)$, $s > 0$ by

$$\begin{align*}
\gamma(s) &= \mathcal{M}(\Delta(S, [s, \infty]; \mathbb{C})), \\
\tau(s) &= \mathcal{M}(\Delta([-1, 0], [s, \infty]; \mathbb{C})),
\end{align*}$$

respectively. These functions are called the Grötzsch modulus and the Teichmüller modulus, respectively. We also use the bounded version of the Grötzsch ring $R^*_G(s) = \mathbb{D}\setminus[0, s)$, where $s \in (0, 1)$. By conformal invariance, the modulus of the path family connecting boundary components of $R^*_G(s)$ is $\gamma(1/s)$.

Grötzsch and Teichmüller moduli functions are strictly decreasing and continuous with range $(0, \infty)$. They are connected by the identity [3, Theorem 8.37]:

$$\Gamma(s) = 2\tau(s^2 - 1), \quad s > 1.$$

For $s > 1$, we also have the identity [3, 8.57]:

$$\gamma(s) = \frac{4}{\pi} \mu(s + 1),$$

where, for $r \in (0, 1)$, $\mu(r)$ is the quantity

$$\mu(r) = \frac{\pi}{2} \frac{K(r')}{K(r)} = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-r^2t^2)}},$$

and $r' = \sqrt{1-r^2}$. We recall that $K(r)$ is referred to as the complete elliptic integral of the first kind.

The function $\mu(r)$ is the conformal modulus of the bounded Grötzsch ring domain $R^*_G(r) = \mathbb{D}\setminus[0, r)$ and hence for $r > 1$,

$$\gamma(r) = \frac{2\pi}{\mu(1/r)}.$$

The following functional identities for $\mu(r)$ are well known [3, 5.2]:

$$\mu(r)\mu(r') = \frac{\pi^2}{4}, \quad \mu(r)\mu\left(\frac{1-r}{1+r}\right) = \frac{\pi^2}{2},$$

and

$$\mu(r) = 2\mu\left(\frac{2\sqrt{r}}{1+r}\right).$$

These relations allow us to obtain useful estimates for the functions $\gamma$ and $\tau$ (see e.g. [3, Chapters 5 and 8]). Canonical ring domains are useful in the theory of quasiconformal mappings because they can be, e.g., used for obtaining lower and upper bounds for the conformal modulus of a given ring domain. For example, we have the following result.
Lemma 9 ([3, Theorem 8.44]). Let $R = R(E, F)$ be a ring in $\mathbb{C}_\infty$ and let $a, b \in E$ and $c, \infty \in F$ be distinct points. Then
\[
\frac{2\pi}{\text{mod}(R)} \geq \tau(s), \quad s = \frac{|a - c|}{|a - b|}.
\]
Equality holds for the Teichmüller ring $R_T(s).

3. Schwarz’ lemma for quasiconformal mappings

In this section we consider the distortion results for quasiconformal mappings. A basic example of such result is the classical Schwarz lemma for analytic functions of the unit disk. This result also useful in studying harmonic mappings, for example the proof the generalization of the Koebe one-quarter theorem for harmonic mappings make use of this inequality.

Lemma 10. Let $f : \mathbb{D} \to f(\mathbb{D}) \subset \mathbb{D}$ be a $K$-quasiconformal mapping with $f(0) = 0$. Then
\[
|f(z)| \leq \varphi_K(|z|),
\]
where $\varphi_K$ is the distortion function
\[
\varphi_K(r) = \mu^{-1}(\mu(r)/K).
\]
Remark 4. For a fixed $K \geq 1$, the bound $\varphi_K(r)$ increases with $r$ from 0 to 1. For $K = 1$ we get the classical Schwarz’ lemma.

3.1 Hyperbolic metrics

The hyperbolic metric is an important tool in contemporary complex analysis. Techniques involving hyperbolic metric are also very useful when studying quasiconformal mappings and geometric function theory in higher dimensions. We start by giving a brief overview to the hyperbolic metric and related results in function theory.

Suppose that $w = f(z)$ is a conformal mapping of the unit disk onto itself. Then by Pick’s lemma we have the equality
\[
\frac{|dw|}{|dz|} = \frac{1 - |w|^2}{1 - |z|^2}.
\]
This identity can be written
\[
\frac{|dw|}{1 - |w|^2} = \frac{|dz|}{1 - |z|^2},
\]
which means that for any regular curve $\gamma$ in the unit disk $\mathbb{D} = \{z : |z| < 1\}$, and for any conformal self mapping $f$ of $\mathbb{D}$, we have
\[
\int_{\gamma} \frac{|dw|}{1 - |w|^2} = \int_{\gamma} \frac{|dz|}{1 - |z|^2}.
\]
We have obtained a length function which is invariant under conformal mappings of the unit disk onto itself. This allows us to give the following definition.

Definition 8. Let $z_1, z_2 \in \mathbb{D}$. Then the hyperbolic distance $\rho$ of $z_1, z_2$ is defined by
\[
\rho_\mathbb{D}(z_1, z_2) = \inf_{\gamma} \int_{\gamma} \frac{2|dz|}{1 - |z|^2},
\]
where the infimum is taken over all regular curves $\gamma$ connecting $z_1$ and $z_2$.

Obviously, the multiplier 2 is harmless, and it is sometimes omitted. By the conformal invariance we can define the hyperbolic distance on any simply connected domain $\Omega \subset \mathbb{C}$ by the formula
\[
\rho_\Omega(z_1, z_2) = \rho_\mathbb{D}(f(z_1), f(z_2)), \quad z_1, z_2 \in \Omega,
\]
where $f : \Omega \to \mathbb{D}$ is a conformal mapping. Existence of such mapping is guaranteed by Riemann’s mapping theorem.

Note that Definition 8 makes sense also in $\mathbb{R}^n$ for all $n \geq 2$, if we consider instead of the unit disk $\mathbb{D}$, the unit ball $\mathbb{B}^n = \{x \in \mathbb{R}^n : |x| < 1\}$. However, for $n \geq 3$ we have few conformal mappings, as by the generalized Liouville theorem (see e.g. [30] and also [31]), every conformal mapping of a domain $\Omega$ in the Euclidean space $\mathbb{R}^n$, $n \geq 3$, is a restriction of a Möbius transformation. Thus, the domains of interest in higher dimensions are essentially the unit ball and the upper half-space $\mathbb{H}^n = \{x = (x_1, \ldots, x_n) : x_n > 0\}$. Because formulas for Möbius transformations of the unit ball $\mathbb{B}^n$ onto $\mathbb{H}^n$ are well known, we arrive at the following definitions.

Definition 9. The hyperbolic distance between $x, y \in \mathbb{B}^n$ is defined by
\[
\rho_\mathbb{B}(x, y) = \inf_{\alpha \in \Gamma_{xy}} \int_{\alpha} \frac{2|dx|}{1 - |x|^2},
\]
and between $x, y \in \mathbb{H}^n$ by
\[
\rho_\mathbb{H}(x, y) = \inf_{\alpha \in \Gamma_{xy}} \int_{\alpha} \frac{|dx|}{x_n},
\]
where $\Gamma_{xy}$ is the family of all rectifiable paths joining $x$ and $y$ in $\mathbb{B}^n$ or $\mathbb{H}^n$, respectively.
It is well-known that the hyperbolic distance $\rho_\Omega$ is a metric in the topological sense, i.e., the following properties hold for $d = \rho_\Omega$:

1. $d(x, y) \geq 0,$
2. $d(x, y) = d(y, x),$ and
3. $d(x, z) \leq d(x, y) + d(y, z),$
4. $d(x, y) = 0,$ if and only if $x = y,$

for all $x, y, z \in \Omega.$ The last condition is the triangle inequality.

A curve $\gamma$ is called a geodesic of the metric $d$ if the triangle inequality holds as an equality for all points $x, y, z$ on $\gamma$ in that order.

One may show that hyperbolic metric always has a geodesic: for any $x, y \in \Omega$ there is a geodesic $\gamma$ connecting $x, y,$ i.e., the infimum in the integral is attained by $\gamma.$ For the ball or half-space those geodesics are always circular arcs orthogonal to the boundary. Further calculations allow us to obtain formulas for $\rho_\mathbb{H}$ and $\rho_\mathbb{B}$ in the closed form. The hyperbolic metrics in the upper half-plane $\mathbb{H}$ and in the unit disk $\mathbb{D}$ are also given by equations

$$\sinh^2 \left( \frac{1}{2} \rho_\mathbb{H}(x, y) \right) = \frac{|x - y|^2}{(1 - |x|^2)(1 - |y|^2)}, \quad x, y \in \mathbb{B}^n,$$
and

$$\cosh \rho_\mathbb{H}(x, y) = 1 + \frac{|x - y|^2}{2 x_n y_n}, \quad x, y \in \mathbb{H}^n.$$

Hyperbolic metric is topologically equivalent to the one defined by the restriction of the Euclidean norm: they define the same open sets. In fact, if $\Omega$ is the unit ball, or its image in a Möbius transformation, the hyperbolic balls

$$\{ y : \rho_\Omega(x, y) < r \}, \quad x \in \Omega, \ r > 0,$$

are Euclidean balls, although their hyperbolic center is not usually the same as the Euclidean center.

There are certain results which make the hyperbolic metric particularly interesting from the point of view of the geometric function theory. For example, many classical results from the Euclidean geometry have hyperbolic analogues, such as the following hyperbolic form of Pythagoras' Theorem:

**Theorem 11 ([4, Theorem 7.11.1]).** *For a hyperbolic triangle with angles $\alpha, \beta, \pi/2$ and corresponding hyperbolic opposite side lengths $a, b, c,$ we have*

$$\cosh c = \cosh a \cosh b.$$

Here the hyperbolic triangle means the domain whose boundary consists of the hyperbolic geodesics connecting three points in the hyperbolic space. In particular, the hyperbolic distance is very useful in complex analysis. For example, we can prove the following version of the classical Schwarz’s lemma (see for example [11, p. 268]). Interestingly, this result does also have a complete analogue for quasiconformal mappings in $\mathbb{R}^n, n \geq 2$ (see [30, 11.2]).

### 3.2 Versions of Schwarz’ lemma involving hyperbolic distance

One may state Schwarz’ lemma by using hyperbolic metric as follows. This version is also called the Schwarz-Pick lemma.

**Lemma 12.** *Let $f : \mathbb{D} \to \mathbb{D}$ be analytic. Then the following inequality holds:*

$$\rho_\mathbb{D}(f(z_1), f(z_2)) \leq \rho_\mathbb{D}(z_1, z_2) \quad \text{for} \ z_1, z_2 \in \mathbb{D},$$

*where the equality holds if and only if $f$ is a Möbius transformation.*

Again, the same idea works for quasiregular mappings of the plane, and in higher dimensions (see [30]) as well. However, there is no obvious way to prove a result of this type for harmonic mappings.

**Lemma 13.** *Let $f : \mathbb{D} \to f(\mathbb{D}) \subset \mathbb{D}$ be a $K$-quasiregular mapping. Then for $z_1, z_2 \in \mathbb{D},$

$$\tan \left( \frac{1}{2} \rho(f(z_1), f(z_2)) \right) \leq K \left( \tan \left( \frac{1}{2} \rho(z_1, z_2) \right) \right).$$

While the hyperbolic metric is a powerful tool in function theory, this approach also has two important limitations:

1. In $\mathbb{R}^n,$ for $n \geq 3,$ the hyperbolic metric cannot be defined for a domain which is not an image of the unit ball by a Möbius transformation.
2. Even in the plane, there are no actual formulas for hyperbolic distance, except in the case of few domains for which the Riemann mapping can be found analytically.

We would like to have a metric which could be defined for any domain of interest and which could be easily computed, or at least estimated. Ideally, we would like this metric to be as similar to the hyperbolic metric as possible in other respects. This leads us to the introduction of quasihyperbolic metric.
3.3 Quasihyperbolic and distance ratio metrics

The quasihyperbolic metric was first introduced by F. W. Gehring and B. P. Palka in 1976 [14], and it has been studied by numerous authors thereafter.

Definition 10. Let $\Omega$ be a proper subdomain of the Euclidean space $\mathbb{R}^n$, $n \geq 2$. We define the quasihyperbolic length of a rectifiable arc $\gamma \subset \Omega$ by

$$\ell_k(\gamma) = \int_{\gamma} \frac{|dz|}{d(z, \partial\Omega)}.$$  

The quasihyperbolic metric is defined by

$$k_\Omega(x, y) = \inf_{\gamma} \ell_k(\gamma),$$

where the infimum is taken over all rectifiable curves in $\Omega$ joining $x$ and $y$. If the domain $\Omega$ is clear from the context we use notation $k$ instead of $k_\Omega$.

Clearly, for $\Omega = \mathbb{H}^n$ the quasihyperbolic metric is the same as the hyperbolic metric, but in the case of the ball it is not. The quasihyperbolic metric of the ball is connected to the hyperbolic metric by the following inequalities (see [31, 3.3]):

$$\rho_{\mathbb{H}^n}(x, y) \leq 2k_{\mathbb{H}^n}(x, y) \leq 2\rho_{\mathbb{H}^n}(x, y), \quad x, y \in \mathbb{H}^n.$$

Obviously, it follows that quasihyperbolic metric is not, in general, invariant under conformal mappings, not even in M"obius transformations. However, we have the following result:

Theorem 14 ([31, 3.10]). If $\Omega$, $\Omega'$ are proper subdomains of $\mathbb{R}^n$ and $f: \Omega \to \Omega' = f(\Omega)$ is a M"obius transformation, then

$$\frac{1}{2}k_\Omega(x, y) \leq k_{\Omega'}(f(x), f(y)) \leq 2k_{\Omega'}(x, y),$$

for all $x, y \in \Omega$.

The proof of the above was first given by Gehring and Palka [14], where a generalization of the result for quasiconformal mappings was also obtained.

In the case of the hyperbolic metric, it is easy to characterize the geodesics of this metric at least in the case of the ball and half-space. The properties, or even existence, of quasihyperbolic geodesics is not immediately clear, and they have been studied by several authors. In $\mathbb{R}^n$ it is, however, true that quasihyperbolic metric is always geodesic. Even this is not true in the more general setting of Banach spaces which we shall not discuss here.

Their results already reveal that the quasihyperbolic metric is useful in study of geometric function theory, but one crucial problem remains. While the quasihyperbolic metric is easy to define, we do not yet have any effective estimates for it. For this reason, we need yet another definition.

Definition 11. The distance ratio metric or $j$-metric in a proper subdomain $\Omega$ of the Euclidean space $\mathbb{R}^n$, $n \geq 2$, is defined by

$$j_\Omega(x, y) = \log \left(1 + \frac{|x - y|}{\min\{d(x), d(y)\}} \right),$$

where $d(x)$ is the Euclidean distance between $x$ and $\partial\Omega$.

Again, if the domain $\Omega$ is clear from the context, we use notation $j$ instead of $j_\Omega$. The distance ratio metric was introduced by F. W. Gehring and B. G. Osgood in 1979 [13]. The above form for the distance ratio metric was introduced by Vuorinen in [29] (see also [30]).

A useful inequality connecting this the $j$-metric to the quasi-hyperbolic metric is the following [14, Lemma 2.1]:

$$k_\Omega(x, y) \geq j_\Omega(x, y), \quad x, y \in \Omega.$$

An inequality in the other direction does not, in general, hold. For example, one may consider the slit plane $\Omega = \mathbb{C}\setminus \mathbb{R}_+$, and two points $s \pm ti$, where $s, t > 0$. When $s \to +\infty$ and $t$ remains fixed, the $j$-metric distance of the points does not change, but the quasihyperbolic distance of the points goes to infinity. If the inequality

$$k_\Omega(x, y) \leq c j_\Omega(x, y), \quad x, y \in \Omega,$$

holds for all $x, y \in \Omega$ where $c \geq 1$ is a constant, then we say that the domain $\Omega$ is uniform. There are several equivalent ways of defining uniformity of a domain. Uniformity of different domains and the respective constants of uniformity have been studied by numerous authors, see e.g. [18,30,31].

4. Two versions of Koebe’s one-quarter theorem for analytic functions

In this section we discuss proofs of Koebe’s one-quarter theorem that based on the arguments using conformal modulus and the extremal property of the Gr"otzsch ring domain.
introduced in the previous section. We start with a short overview of this topic. For a thorough discussion about conformal modulus, we refer to the book of Lehto and Virtanen [17]. See also [22].

### 4.1 Grötzsch module theorem and Koebe’s one-quarter theorem

Next we will consider the extremal problem of the following type: Among all ring domains which separate two closed connected sets \( E_1, E_2 \), find one whose modulus has the greatest value. Because of the conformal invariance of the problem, we may normalize the situation so that \( E_1 \) is the unit circle \( S = \{ z : |z| = 1 \} \), and \( E_2 \) contains two points 0 and \( r \in (0, 1) \). From the normalization above, it is obvious that the solution of the extremal problem is the domain whose boundary consists of the circle \( S \) and the line segment \([0, r] \). This domain, denoted by \( R_G(r) \), is called Grötzsch’s extremal domain. Its modulus is \( \operatorname{mod}(R_G(r)) = \mu(r) \) which was considered in the previous section. Recall that the function \( \mu : (0, 1) \rightarrow (0, \infty) \) is a decreasing homeomorphism.

**Theorem 15 (Grötzsch’s module theorem [17, p.54]).** If the ring domain \( \Omega \) separates the points 0, 1 from the unit circle \( S \), then

\[
\operatorname{mod}(\Omega) \leq \mu(r).
\]

We denote the complex plane cut along \([R, +\infty)\), where \( R \geq 0 \) by \( \mathbb{C}^R \), and let \( \mathbb{C}^R_1 = \mathbb{C}^R \setminus \mathbb{D}^R \). Note that \( \mathbb{C}_1^R \) is the image of the Grötzsch’s extremal domain \( R_G(r) \) in the inversion \( z \mapsto 1/z \).

In order to prove the next result, which has some other classical proof using Bieberbach theorem, we need to recall following: One of the classical problems for the class \( \mathcal{S} \) of analytic univalent functions, normalized so that

\[
f(z) = z + a_2 z^2 + a_3 z^3 + \cdots, \quad |z| < 1,
\]

was to find the sharp bound for the second coefficient \( a_2 \). The solution to this problem, \( |a_2| \leq 2 \), was obtained by Bieberbach and this fact plays a crucial role in the classical function theory, e.g. in the proof of the Koebe one-quarter theorem. It is worth remarking that in the analytic case, the Bieberbach conjecture \(|a_n| \leq n\) for each \( f \in \mathcal{S} \) has been proved by de Branges [6].

**Theorem 16 (Koebe’s one-quarter theorem).** If \( f \in \mathcal{S} \) then \( f(\mathbb{D}) \) contains the disk \( \mathbb{D}_{1/4} \).

**Proof.** Suppose that \( f \in \mathcal{S} \), and let \( r \in (0, 1) \). Write \( D_r = \mathbb{D} \setminus \mathbb{D}_r \) and let \( G_r = f(D_r) \). Suppose that \( \delta = \delta_f = \operatorname{dist}(0, \partial f(\mathbb{D})) \). From the conformal invariance of the modulus it follows that \( \operatorname{mod}(G_r) = \delta^2 \).

For \( r \in (0, 1) \), let \( r_* = \min\{|f(z)| : |z| = r\} \). Then by the monotonicity of the conformal modulus and \([3, 8.60]\),

\[
\operatorname{mod}(C^R_1) < \log \frac{2\delta(1 + \sqrt{1 - r^2})}{r}.
\]

By the monotonicity of the conformal modulus and Grötzsch’s theorem, \( \operatorname{mod}(D_r) \geq \operatorname{mod}(C^R_1) \) and hence, because \( r_* = r + o(r) \), we have

\[
\log \frac{1}{r} \leq \log \frac{4\delta}{r + o(r)}.
\]

By letting \( r \rightarrow 0 \), it follows \( \delta_f \geq 1/4 \), and thus \( \mathbb{D}_{1/4} \subset f(\mathbb{D}) \).

### 4.2 Koebe one-quarter theorem for analytic functions

**Example 3.** The function \( f_n(z) = e^{i\pi/n} \) shows that, for \( f \) an analytic function on the unit disk \( \mathbb{D} \), with \( f(0) = 0 \) and \( |f'(0)| \geq 1 \), there is no absolute constant \( r \) such that the disk \( \mathbb{D}_r \) is a subset of the image set \( f(\mathbb{D}) \).

This example shows that the hypothesis that \( f \) is univalent is essential for the validity of Koebe’s one-quarter theorem. It is a natural question to ask whether there exists an appropriate generalization of this theorem to the functions which are not injective.

The following very simple proof of a result which can be considered to be an appropriate generalization of Koebe one-quarter theorem (with the best constant 1/4) is due to Mateljević [20]. In order to state the theorem we first introduce some notations.

If \( z \) and \( w \) are complex numbers, by \( \Lambda(z, w) \) we denote the half-line \( \Lambda(z, w) = \{ z + \rho(w - z) : \rho \geq 0 \} \). We also write \( \Lambda(w) = \Lambda(0, w) \). For \( \theta \in \mathbb{R} \) we use the shorthand notations \( \Lambda_\theta \) and \( \Lambda_\theta^w \), respectively for \( \Lambda(0, e^{i\theta}) = \{ \rho e^{i\theta} : \rho \geq 0 \} \) and \( \Lambda(a, a + e^{i\theta}) \). By \( \omega = \omega_f \) we denote the modulus of continuity of the function \( f \).

**Lemma 17 (Koebe lemma for analytic functions).** Suppose that \( f \) is an analytic function on the closed unit disk
\( \mathbb{D} \), \( f(0) = 0 \) and \( |f'(0)| \geq 1 \). Then for every \( \theta \in \mathbb{R} \) there exists a point \( w \) on the half-line \( \Lambda \theta \) which is in \( f(\mathbb{D}) \) such that \( |w| \geq 1/4 \).

**Proof.** Suppose that there exists a \( \theta \) for which the theorem is not true. Without loss of generality we may assume that the image set \( f(\mathbb{D}) \) does not intersect with \( \Lambda = [1/4, \infty) \). Then Koebe function defined by

\[
k(z) = \frac{z}{1 + z^2},
\]
maps \( \mathbb{D} \) onto the domain \( \mathbb{C}\setminus \Lambda \). As \( f \) is subordinate to \( k \), by the subordination principle, it follows that \( |f'(0)| \leq 1 \).

If \( |f'(0)| = 1 \), then \( f = k \) and \( f(1) = k(1) = 1/4 \), which is a contradiction, because it was assumed that \( f(\mathbb{D}) \) does not intersect with \( \Lambda = [1/4, \infty) \). Hence \( |f'(0)| < 1 \), which is again a contradiction. \( \square \)

As an application of Lemma 17, we immediately obtain the following result, which we call the first version of Koebe theorem for analytic functions.

**Theorem 18 (The first version of Koebe theorem for analytic functions).** Let \( B = \mathbb{D}_r(a), \) and suppose that \( f \) is an analytic function on \( B \), \( D = f(B) \), and \( f(a) = b \). Then for every \( \theta \in \mathbb{R} \) and \( b \in D \), there exists a point \( w \) on the half-line \( \Lambda^0_\theta \), which is in \( f(B) \), such that \( |w - b| \geq \rho_f(a) \), where

\[
\rho_f(a) = \frac{1}{4} |f'(a)|.
\]

In particular, there exists a point \( w \in D \) such that \( |w| - |b| = |w - b| \geq \rho_f(a) \).

**Proof.** If \( f'(a) \neq 0 \), then apply Lemma 17 to the function \( s(f(a + rz) - b) \), where \( s = \frac{1}{rf'(a)} \).

The result follows. \( \square \)

The first version of Koebe theorem for analytic functions has a connection to the Bloch theorem. The Bloch theorem states the following.

**Theorem 19 (Bloch theorem).** If \( f \) is a holomorphic function on \( \mathbb{D} \), normalized so that \( |f'(0)| \geq 1 \), then \( f(\mathbb{D}) \) contains a disk of radius greater than an absolute constant, so-called Bloch constant.

This gives a version of our key estimate with an absolute constant.

Suppose that the complement of \( D = f(\mathbb{D}) \) is not empty, and let \( d = d(w) = \text{dist}(w, \partial D) \). Then the Bloch theorem states that there is an absolute constant \( c \) and \( w_0 \in D \) such that the disk \( B = \mathbb{D}_r(w_0) \) of radius \( R = c|f'(a)| \) is contained in \( D \). It is clear that there is a point \( w_1 \) in the intersection of \( D \) with the line \( \Lambda \) which contains points 0 and \( w_0 \) such that \( |w_1| - |w_0| = |w_1 - w_0| \geq R \). Hence, \( R \leq \omega_f(r) \). Thus, under conditions of Theorem 18, we have

\[
|rf'(a)| \leq \omega_f(r).
\]

Here, by \( \omega = \omega_f \) we again denote the modulus of continuity of \( f \). By using Koebe’s one-quarter theorem for univalent functions we may then generalize the first version of Koebe theorem for non-univalent analytic functions (Theorem 18).

**Theorem 20 (Second version of Koebe theorem for analytic functions).** Let \( B = \mathbb{D}_r(a), \) and let \( f \) be an analytic function on \( B \), \( D = f(B) \) and \( f(a) = b \). Suppose that the unbounded component \( D_\infty \) of \( D \) is not empty and write \( d_\infty = d_\infty(b) = \text{dist}(b, D_\infty) \). Then

(1) \( r|f'(a)| \leq 4d_\infty \), and

(2) If \( D \) is simply connected, then \( D \) contains the disk \( \mathbb{D}_r(b) \), where \( \rho = \rho_f(a) = r|f'(a)|/4 \).

**Proof.** Let \( D_0 \) be the complement of \( \overline{D_\infty} \). Suppose that \( \psi \) is a conformal mapping of \( D_0 \) onto \( \mathbb{D} \) such that \( \psi(b) = 0 \), and let \( \psi = \psi^{-1} \). Write \( F = \psi \circ f \). Then, by the Koebe theorem for univalent function, the disk \( B_\ast = \mathbb{D}_{R_0}(b) \), where \( R_0 = |\psi'(0)|/4 \), is contained in \( D_0 \), and hence \( R_0 \leq d = \text{dist}(b, \partial D_0) \). It follows that

\[
|\psi'(0)| \leq 4d.
\]

Then, by the Schwarz lemma,

\[
|F'(a)| \leq 1.
\]

Because \( \psi'(b) \psi'(0) = 1 \), we have \( F'(a) = \psi'(b) f'(a) = f'(a)/\psi'(0) \). Hence it follows, by (4), that \( |f'(a)| \leq |\psi'(0)| \).

Therefore, since \( d = d_\infty \), by (3) we have \( |f'(a)| \leq 4d_\infty \), which proves the part (1).

If \( D \) is simply connected, then

\[
d_\infty = d_\infty(b) = \text{dist}(b, D_\infty) = d = \text{dist}(b, \partial D).
\]

This proves the part (2). \( \square \)
Suppose, in addition, that $f$ is holomorphic on $\overline{B}$. Then for every $\theta \in \mathbb{R}$ and $b \in D$, there exists a point $w$ on the half-line $\Lambda^+_{\theta}$, which is in $f(B)$ such that
$$|w - b| \geq \frac{1}{4} |r| f'(\alpha)|.$$ Thus, Theorem 18 is a special case of Theorem 2.

Note that, in general, $D = f(B)$ is not a simply-connected domain and the disk $B_\ast = B(b, R)$ does not belong to $D$ (see Examples 4 and 5 below).

**Example 4.** Let $Q = Q_r = (-r, r)^2$ for $r > 2\sqrt{2}\pi$, and let $\phi(z) = \exp(z)$. Then $\phi(0) = \phi(0) = 1$ and $\phi(Q) = D$, where $D = D_r = \{w: \exp(-r) < |w| < \exp(r)\}$ is a doubly-connected ring domain. Then by the second version of the Koebe theorem for analytic function, Theorem 2, we have $|\phi(0)| = r \leq 4d$, where $d = d_\infty(1)$. Because $Q_r/\mathbb{Z} \subset \Delta_r$, it is easy to see that $\exp(r/\sqrt{2}) - 1 \leq d_\infty(1)$. Hence, $d_\infty(1) \to +\infty$ when $r \to +\infty$. However, the disk $D_0(1)$ is not a subset of $D = D_r$ for any $r > 0$.

### 5. Koebe one-quarter theorem for harmonic univalent functions

In this section, we consider generalizations of the Koebe one-quarter theorem for harmonic univalent mappings. Our exposition is mainly based on [8]. Denote by $S_H$ the class of all complex-valued harmonic univalent and sense-preserving mappings $f = h + \bar{g}$ in the unit disk $\mathbb{D}$ with $f(0) = 0 = f_\bar{z}(0) - 1$. We note that $S_H$ reduces to $S$, the class of normalized univalent analytic functions in $\mathbb{D}$ whenever the coanalytic part of $f$ is zero, i.e., $g(z) \equiv 0$ in $\mathbb{D}$. Furthermore, for $f = h + \bar{g} \in S_H$ with $g'(0) = b$ and $|b| < 1$ (because $J_f(0) = 1 - |g'(0)|^2 = 1 - |b|^2 > 0$), the function
$$F = \frac{f - \overline{b}f}{1 - |b|^2}$$
is also in $S_H$. We observe that $F_\bar{z}(0) = 0$, and this function is obtained by applying an affine mapping to $f$. Thus, we may sometimes restrict our attention to the subclass
$$S^{0}_H = \{ f \in S_H : f_\bar{z}(0) = 0 \}.$$

Also, we note that the condition $f_\bar{z}(0) = 0$ is equivalent to the assertion that the second complex dilatation $\omega(z)$ (see also Theorem 2) of $f$ is zero at the origin. Clearly,
$$S \subset S^{0}_H \subset S_H.$$ Although both $S_H$ and $S^{0}_H$ are known to be normal families, only $S^{0}_H$ is compact with respect to the topology of locally uniform convergence (see [7,8]).

**Example 5.** Consider $f(z) = z + \frac{z^n}{n}z^{n-2}$ for $n \geq 3$. Note that
$$f_\bar{z}(z) = 1 + \left(\frac{n-1}{n}\right)z^{n-2}, \quad f_\bar{z}(z) = \frac{z^{n-1}}{n}$$
and
$$f_\bar{z}(z) = \left(\frac{n-1}{n}\right)z^{n-2}.$$

Thus, $f$ is not harmonic in $\mathbb{D}$. However, $\nabla f$, the Laplacian of $f$ is analytic in $\mathbb{D}$ and hence, $f$ is biharmonic in $\mathbb{D}$. Now we consider the cases $n = 3$ and $n > 3$ separately. For $n = 3$, $f \neq 0$ in $\mathbb{D}$ and
$$\text{Re } f_\bar{z}(z) > |f_\bar{z}(z)|$$
is seen to be equivalent to
$$\text{Re } \left(1 + \frac{2}{3}|z|^2\right) > \frac{|z|^2}{3}, \quad \text{i.e. } 1 + \frac{|z|^2}{3} > 0$$
showing that $f$ is univalent and sense-preserving in $\mathbb{D}$. Moreover,
$$\left|\frac{f_\bar{z}(z)}{f_\bar{z}(z)}\right| = \frac{|z|^2}{3 + 2|z|^2} < \frac{1}{5}.$$ Consequently, $f$ is a quasiconformal map.

Next, for $n > 3$, we see that $\text{Re } f_\bar{z}(z) > |f_\bar{z}(z)|$ is equivalent to
$$\text{Re } \left(1 + \frac{n-1}{n}\right)z^{n-2} > \left|\frac{z^{n-1}}{n}\right|, \quad z \in \mathbb{D}$$
which holds, because the inequality
$$1 - \left(\frac{n-1}{n}\right)|z|^{n-1} > \left|\frac{z^{n-1}}{n}\right|, \quad \text{i.e. } 1 > |z|^{n-1},$$
holds in $\mathbb{D}$. Thus, $f$ is univalent and sense-preserving in $\mathbb{D}$ for $n \geq 3$. In Figures 3–6, the domain $f(\mathbb{D})$ is drawn for $n = 4, 5, 6, 7$.

**Theorem 21.** Each function $f \in S^{0}_H$ satisfies the inequality
$$|f(z)| \geq \frac{1}{4} \frac{|z|}{(1 + |z|)^2}, \quad |z| < 1.$$ In particular, the range of $f$ contains the disk $|w| < 1/16$.\"
Proof. Suppose that $F \in S^H_0$ and $0 < a < 1$, and write $f(z) = F(az)/a$. Then $f \in S^H_0$ and $f$ is a homeomorphism of the closed unit disk $\bar{D}$. Let $\Omega = f(\bar{D})$, and let $\Omega_\varepsilon$ be the ring domain $\Omega_\varepsilon = \Omega \setminus \{w : |w| \leq \varepsilon\}$. Denote by $\delta$ the distance from the origin to the boundary curve $\partial \Omega$. We may assume that $\delta \in \partial \Omega$. Let $\Delta$ be the Grötzsch ring $\Delta = \mathbb{C} \setminus ([\delta, \infty) \cup \{w : |w| \leq \varepsilon\})$. Then the well-known symmetrization argument gives the inequality $\text{mod}(\Delta) \leq \text{mod}(\Omega_\varepsilon)$. We also have the estimate [30, p.67]:

$$\text{mod}(\Delta) \geq \frac{2\pi}{\log(4\delta/\varepsilon)}.$$

Fix $\beta > 1$ and observe that $|f(z)| = \beta \varepsilon + O(\varepsilon^2) > \varepsilon$ on the circle $|z| = \beta \varepsilon$ for sufficiently small $\varepsilon > 0$. Let $\rho$ be admissible for the ring domain $A(\beta \varepsilon) = \{z : \beta \varepsilon < |z| < 1\}$, and define $\rho(z) = 0$ outside $A(\beta \varepsilon)$. We define $\tilde{\rho}$ on $\Omega_\varepsilon$ by

$$\tilde{\rho}(w) = \frac{\rho(z)}{|h'(z)| - |g'(z)|}, \quad w = f(z)$$

where $f = h + \bar{g}$.

Let $\tilde{\gamma}$ be a curve connecting the boundary components of $\Omega_\varepsilon$, and let $\gamma$ be the restriction of its preimage $f^{-1}(\tilde{\gamma})$ to a curve connecting boundary components of $A(\beta \varepsilon)$. Then

$$\int_{\tilde{\gamma}} \tilde{\rho}(w) |dw| = \int_{\gamma} \frac{\rho(z)|df/\partial s(z)|}{|h'(z)| - |g'(z)|} |dz|,$$

Mathematics Newsletter

Vol. 21 #3, December 2011
where \(\partial f/\partial s\) denotes the directional derivative of \(f\) along \(\gamma\).

Since

\[
\left| \frac{\partial f}{\partial s}(z) \right| \geq |h'(z)| - |g'(z)|,
\]

we have

\[
\int_{\gamma} \tilde{\phi}(w) \, |dw| \geq \int_{\gamma} \rho(z) \, |dz| \geq 1,
\]

and hence \(\tilde{\phi}\) is admissible for \(\Omega_e\).

We note that \(f(z) = F(az)/a\) for some \(a \in (0, 1)\), and thus by the Schwarz lemma

\[
\frac{J_f(z)}{|h'(z)| - |g'(z)|^2} \leq 1 + \left| \frac{\omega(z)}{1 - a|z|} \right|,
\]

where \(\omega = h'/g'\) is the complex dilatation of \(f\). It follows that

\[
\text{mod}(\Omega_e) \leq \iint_{A(\beta \varepsilon)} \frac{1 + a|z|}{1 - a|z|} \rho(z)^2 \, dx \, dy,
\]

for any \(\rho\) admissible for \(A(\beta \varepsilon)\). We choose the admissible function

\[
\rho = \frac{1 - ar}{r(1 + ar)} \left[ \int_{\varepsilon}^{1} \frac{1 - at}{(1 + at)^{1 - r}} \, dt \right]^{-1}, \quad r = |z|.
\]

Then

\[
\text{mod}(\Omega_e) \leq 2\pi \left[ 2 \log \frac{1 + a\beta \varepsilon}{1 + a} - \log(a\beta \varepsilon) \right]^{-1}.
\]

From this and the estimate for the Grötzsch ring, we obtain

\[
2 \log \frac{1 + a\beta \varepsilon}{1 + a} - \log(a\beta \varepsilon) \leq \log \frac{4\delta}{\varepsilon}.
\]

When \(\varepsilon \to 0\) we get

\[
\delta \geq \frac{1}{4\beta(1 + a)^2} \to \frac{1}{(1 + a)^2} \quad \text{as} \quad \beta \to 1.
\]

Because \(|f(z)| \geq \delta\) on the unit circle, we have

\[
|F(az)| \geq \frac{a}{4(1 + a)^2} \quad \text{for} \quad |z| = 1,
\]

proving the claim. \(\blacksquare\)

References


1. A Gateway to Modern Mathematics: Adventures in Iteration-I by S. A. Shirali

Resume of “Adventures in Iteration”

To “reiterate” anything means to say something for a second time. In mathematics, “iteration” refers to any sequence of operations that is performed repeatedly; its essential aspect is that the “output” at each stage is used as the “input” for the following step. Many actions in elementary arithmetic and algebra are iterations in hidden form; e.g., the Euclidean algorithm for finding the greatest common divisor of two integers, the division algorithm for finding the square root of a number, Newton’s method for the numerical solution of equations, and the simplex method for solving linear programming problems.

Iterations are an exciting topic to study, for the amateur as well as the professional. Many of the iterations in elementary mathematics offer scope for extended investigation; e.g., the Kaprekar iteration that leads to the number 6174. Another example is the “four-numbers iteration.” In the process, one encounters unsolved problems that look easy but are extremely difficult, e.g., the Collatz “$3n + 1$” problem. Concepts such as those of fixed point, limit point, convergence, orbit, etc., emerge naturally while studying iterations. They are like a gateway for learning important themes of modern mathematics, such as fractals and chaos; they offer a route for experiencing the experimental and visually aesthetic side of mathematics.
mathematics. The Mandelbrot set, the snowflake curve and
the Sierpinski triangle, for example, are all defined through
iterations.

These topics, and many more, are studied in the two-volume
book *Adventures in Iteration*. Volume 1 is at an elementary
level, and is suitable for students aged 13–18 years; Volume 2 is
more challenging, and suitable for students aged 15–20 years.
Teachers who run mathematics clubs in their schools will find
here a rich source of material. The book will also be of value
to members of the general public who have an interest in mathe-
matics and take delight in learning a beautiful area of modern
mathematics.

Reviewer’s Comments: “I enjoyed reading this manuscript,
and I think that anyone with any curiosity about numbers
and their many strange and wonderful properties will find it
fascinating too. Writing a book around the topic of ‘itera-
tion’ is an excellent idea – it is a central theme of current
research in mathematics (under such names as ‘non-linear
systems theory’, ‘chaos’) yet there are examples easily acces-
sible to non-mathematicians, and easily stated problems which
have resisted solution so far. The author’s treatment conveys
very effectively the ‘culture’ of mathematics – the ideas which
interest mathematicians and have done world-wide over the
millennia.” a reviewer.

Books Under Publication

**Title:** “Mathematics 12–14” in 4 volumes by S. A. Shirali. The
first volume of the book is being printed.

The book is intended for students in the age group of 12 to
14 years in schools, however it would be useful also to students
in 11th and 12th standards and of course teachers.

A reviewer writes “Shirali’s book is a superb introduction
to some substantial material for students of mathematics in
classes 7, 8 and 9 in Indian schools. If this book had come
twenty years ago, there would have been an entire generation
of school students who would have “journeyed into a world
of pattern power and beauty,” to quote the author. This book
is far from being routine and needs dedication both from the
teacher and the pupil. It is time that such a book is available
at a reasonable price to our students. For those students who
have missed reading such a book at an earlier level, it will
serve as revision material even at the plus two level. I would
like to add that these volumes will inspire students to seek
a career in pure or applied mathematics. Any publisher
who offers to publish this will be doing a great service to
Indian Mathematics and will also find a large market from
the enormous number of mathematically talented students in
Indian schools. – Alladi Sitaram”

**Ramanujan Series on Little Mathematical Treasures**

**Editorial Policy**

In times past, mathematics was studied and researched by
a select few – an elite class who valued the subject for
its intellectual elegance and perhaps also for its philosophic
connotations. But today it is studied and practiced by an
extremely large number of individuals, for mathematics has
acquired a pivotal role in the sustenance of a technologically
advanced society.

Broadly speaking, there are two distinct groups of
students to whom expository material in mathematics may be
addressed:

(a) students who have a deep interest in mathematics and
wish to pursue a career involving research and teaching of
mathematics;

(b) students who are good at mathematics but wish to pursue
a career in some other discipline.

Clearly such students require suitable material (in addition to
contact with inspiring teachers and practicing mathematicians)
to nurture their talent in the subject.

To this list we may add professionals in industry and govern-
ment and teachers in mathematics and other disciplines who
require a good understanding of certain areas of mathematics
for their professional work.

The proposed series is addressed to mathematically mature
readers and to bright students in their last two years of
school education. It is envisaged that the books will con-
tain expository material not generally included in standard
school or college texts. New developments in mathematics
will be presented attractively using mathematical tools famil-
iliar at the high school and undergraduate levels. There will be
problem sets scattered through the texts, which will serve to
draw the reader into a closer hands-on study of the subject.
Readers will be invited to grapple with the subject, and so
experience the creative joy of discovery and contact with beauty.

A thing of beauty is a joy forever . . . . So it is with mathematics. The discoveries of Archimedes, Apollonius and Diophantus have been sources of joy from very ancient times. It is our hope that these books will serve our readers in a similar manner.

Proposals for publication of a book in the series is invited either with a note for writing a book for the series or with a draft of the proposed book. Decision of the Editorial Board will be final.

The RMS series on LMT is jointly published by RMS and Universities Press. The authors are paid a royalty of 10%. The books are distributed in India and abroad by the Universities Press.

Editorial Board:
K. R. Parthasarathy
Phoolan Prasad (Editor-in Chief)
E. Sampathkumar (Managing Editor)
V. P. Saxena

Address of Correspondence:
Prof. Phoolan Prasad
Department of Mathematics, Indian Institute of Science
Bangalore 560 012
E-mail: prasad@math.iisc.ernet.in

Workshop on Recent Trends on Partial Differential Equations and Applications-2012

18–19 March, 2012

Venue: University of Hyderabad, India.

Description: The National Workshop on “Recent Trends on Partial Differential Equations and Applications” is being organized with the objective to provide a forum for young researchers to learn research methodologies and interact with experts. The focus of the activity will be on Partial Differential Equations, Fluid Dynamics and other related areas. The participating candidates may be in any area that is close to the theme of the workshop. The number of out station participants is limited. All the participants will be provided three tier AC train fare, accommodation and local hospitality. The Workshop on Recent Trends in Partial Differential Equations and Applications – 2012 will focus on:

(1) Expert lectures across various areas close to the theme
(2) Interactive sessions with experts for discussions, suggestions.

Important Dates:
Last date for receiving applications : 1 February 2012
Information to short listed candidates : 15 February 2012
The participants applying for the workshop can also mail their application to
bs.padmavathi@gmail.com
mailto:bs.padmavathi@gmail.com

to reach before 1 February, 2012.

For Further Information Contact/Visit:
B. Sri Padmavati
Department of Mathematics and Statistics
University of Hyderabad
Hyderabad 500 046
Phone: 040-23134201, 040-23134051
Fax: 040-23011469
E-mail: bs.padmavathi@gmail.com
mailto:bs.padmavathi@gmail.com
bspsm@uohyd.ernet.in
mailto:bspsm@uohyd.ernet.in


The 8th International Conference on Scientific Computing and Applications (SCA2012)

1–4 April, 2012

Venue: University of Nevada Las Vegas (UNLV), Las Vegas, Nevada.

Description: This will be the 8th of the sequences of conferences on Scientific Computing and Applications (SCA) held in the Pacific Rim region (held previously in China, Canada, Hong Kong, Korean). This is the first time to be held in USA. The purpose of the meeting is to provide a forum for researchers working on various aspects of Scientific
Computing and Applications to meet and move this area forward.

For Further Information Visit:

Aim Workshop: Vector Equilibrium Problems and their Applications to Random Matrix Models

2–6 April, 2012


Description: This workshop, sponsored by AIM and the NSF, will be devoted to the study of vector equilibrium problems and their application to the asymptotic analysis of random matrix models.

For Further Information Visit:
http://www.aimath.org/ARCC/workshops/vectorequilib.html

Spring School in Probability

23–27 April, 2012

Venue: Inter-University Center, Dubrovnik, Croatia.

Description: The school is primarily aimed at advanced doctoral students and early postdocs in probability theory and random processes. There will be five intensive courses broadly focused on random walks and jump processes and their applications to real-world problems. More specifically, the topics covered will include the coupling of Lévy processes, probabilistic potential theory of jump processes and heat kernel estimates, random walks on graphs and disordered media and their scaling limits, reinforced random walks, and discretization of jump processes and Lévy driven stochastic differential equations. Participants will have a chance to present their research results.

For Further Information Visit:
http://web.math.hr/ssp-iuc

International Conference on “Applied Mathematics and Approximation Theory 2012”

17–19 May, 2012

Venue: TOBB University of Economics and Technology, Ankara, Turkey.

Topics: Applied Mathematics and Approximation Theory in the broad sense.

For Further Information Visit:
http://amat2012.etu.edu.tr/

7th European Conference on Elliptic and Parabolic Problems

20–25 May, 2012

Venue: Gaeta, Italy.

Description: Besides Elliptic and Parabolic issues, the topics of the conference include Geometry, Free Boundary Problems, Fluid Mechanics, Evolution Problems in general, Calculus of Variations, Homogenization, Control, Modeling and Numerical Analysis. In addition to the plenary talks parallel sessions and minisymposia will be organized.

For Further Information Visit:
http://www.math.uzh.ch/gaeta2012

Aim Workshop: Contact Topology in Higher Dimensions

21–25 May, 2012


Description: This workshop, sponsored by AIM and the NSF, will be devoted to developing high dimensional contact topology.

For Further Information Visit:
http://www.aimath.org/ARCC/workshops/contacttop.html
International Conference “Theory of Approximation of Functions and its Applications”

28 May – 3 June, 2012

Venue: Kamianets-Podilsky Ivan Ohienko National University, Kamianets-Podilsky, Ukraine.

Description: International Conference “Theory of Approximation of Functions and its Applications” dedicated to the 70th anniversary of corresponding member of National Academy of Sciences of Ukraine, Prof. O. I. Stepanets (1942–2007).

For Further Information Visit:

Clay Mathematics Institute 2012 Summer School “The Resolution of Singular Algebraic Varieties”

3–30 June, 2012

Venue: Obergurgl, Tyrolean Alps, Austria.

Description: The resolution of singularities is one of the major topics in algebraic geometry. Due to its difficulty and complexity, as well as certain historical reasons, research to date in the field has been pursued by a relatively small group of mathematicians. However, the field has begun a renaissance over the last twenty years. This school will consist of three weeks of foundational courses supplemented by exercise and problem sessions, designed to provide graduate students and young mathematicians with a comprehensive framework for research in this field. The fourth week will consist of mini-courses with selected experts, aimed at providing participants with state of the art techniques, as well as a survey of some of the main open problems and the most promising approaches now under investigation. Facilities will be provided for lectures, meals and lodging at the Obergurgl Center.

Deadline: For applications is February 1, 2012.

For Further Information Visit:
http://www.claymath.org/summerschool

Noncommutative Algebraic Geometry

18–29 June, 2012

Venue: Mathematical Sciences Research Institute, Berkeley, California.

Description: This workshop will introduce some of the major themes of the MSRI program “Interactions between Noncommutative Algebra, Representation Theory, and Algebraic Geometry” to be held in the spring of 2013. There will be four mini-courses on the topics of noncommutative projective geometry, deformation theory, noncommutative resolutions of singularities, and symplectic reflection algebras. As well as providing theoretical background, the workshop will aim to equip participants with some intuition for the many open problems in this area through worked examples and experimental computer calculations.

For Further Information Visit:
http://www.msri.org/web/msri/scientific/workshops/summer-graduate-workshops/show/-/event/Wm9213

Random Matrix Theory and its Applications II

18 June – 15 August, 2012

Venue: Institute for Mathematical Sciences, National University of Singapore, Singapore.

For Further Information Visit:
http://www2.ims.nus.edu.sg/Programs/012random/index.php

Carisma IIMC Workshop 2012
Optimization Methods & Risk Analysis: Applications in Finance

14–17 March, 2012


Workshops:


Part II: Stochastic Optimization Models and Solution Methods 16–17 March 2012

Part III: News Analytics 17 March 2012

For More Details Download the Above CARISMA-IIMC Brochure 2012 From:
http://www.iimcal.ac.in/carisma-iimc-workshop-2012

2nd National Conference on Mathematical Analysis and its Applications (NCMAA12)

5–6 April, 2012

Organized by Department of Mathematics, Tripura University.

For Further Details Visit:
http://www.tripurauniv.in

For Further Details and Call for Abstracts, Mail to One of:
anjam_2002@yahoo.co.in;
halder_731@rediffmail.com;
subrata.bhowmik.math@rediffmail.com

or write to:
Prof. A. Mukherjee or
Dr. S. Bhattacharya (Halder) or
Sri. S. Bhowmik at
Dept. of Mathematics
Tripura University, Tripura 799 022

International Symposium on Orthogonal Polynomials and Special Functions – A Complex Analytic Perspective

11–15 June, 2012

Aim and Scope: The areas of orthogonal polynomials and special functions have many applications in other branches of mathematics. On the other hand many different tools are used to attack fundamental questions in the fields of orthogonal polynomials and special functions and among these tools complex analysis plays a fundamental role. The symposium will focus on applications of complex analysis in orthogonal polynomials and special functions. It is the aim to bring together scientists working in orthogonal polynomials and special functions and in complex analysis.


Symposium Venue: The Royal Danish Academy of Sciences and Letters in Copenhagen.

For Further Details and Deadlines Visit:
http://www.matdat.life.ku.dk/~henrikp/osca2012/

International Conference on Complex Analysis and Related Topics The 13th Romanian-Finnish Seminar

26–30 June, 2012

Description: The program is divided into sections as follows:

1. Geometric function theory and classical complex analysis;
2. Quasiconformal mappings, Teichmüller spaces, metric geometry and related topics in classical analysis;
3. Potential theory and PDEs, geometric analysis, complex geometry, functional analytical methods in complex analysis.
There will be invited and contributed lectures. A proceedings volume of the conference will be published; for the proceedings of last edition see

http://www.springerlink.com/content/v7420542w1j3/

For Further Details Visit:

The 6th International Conference ‘Inverse Problems: Modeling and Simulation’

21–26 May, 2012

Venue: Antalya, Turkey.

Aim: The main aim of the conference is to bring together all classical and new inverse problems areas from various international scientific schools, and to discuss new challenges of inverse problems in current interdisciplinary sciences. All these International Conferences were organizing under the auspices of the leading international journals “Inverse Problems”, “Inverse Problems in Science and Engineering” and “Inverse and Ill-Posed Problems”.

For Further Details Visit:
http://www.ipms-conference.org/index.html

The International Conference on the Frontier of Computational and Applied Mathematics: Tony Chan’s 60th Birthday Conference

08–10 June, 2012

Venue: Institute for Pure and Applied Mathematics (IPAM), United States

Aim: The aim of this conference is to provide a forum to discuss recent developments and future directions in applied and computational mathematics. It will also provide an opportunity to get leading experts as well as junior researchers together to exchange and stimulate new ideas from a wide spectrum of disciplines. The program of the conference includes invited talks, panel discussion and poster session.

For Further Details Visit:
http://www.ipam.ucla.edu/programs/chan2012/

Workshop on Partial Differential Equations, Harmonic Analysis, Complex Analysis, and Geometric Measure Theory

17–23 June, 2012

Organizers: Dorina Mitrea, University of Missouri at Columbia Irina Mitrea, Temple University Katharine Ott, University of Kentucky

Aim: This workshop focuses on research problems at the interface between Complex Analysis, Harmonic Analysis, PDE, and Geometric Measure Theory. This choice is motivated by the fact that combinations of techniques originating in these fields have proved to be extremely potent when dealing with a host of difficult and important problems in analysis. Topics to be analyzed at the workshop include boundary value problems for single equations and systems of equations, the Poisson kernel for elliptic systems, and the boundary point principle for the Laplacian.

For Further Details Visit:
http://www.ams.org/programs/research-communities/mrc-12

International Center for Theoretical Sciences (ICTS)

Tata Institute of Fundamental Research

The International Centre for Theoretical Sciences (ICTS) and the Centre for Applicable Mathematics (CAM) of the Tata Institute of Fundamental Research hereby invite applications for one position of “Scientific Outreach Coordinator” for “Mathematics of the Planet Earth (MPE) 2013 – India.” The position will be at ICTS and CAM, Bangalore.

Nature of Job: The year 2013 has been marked for a wide variety of mathematical activities under a very broad umbrella of initiatives, called the Mathematics of the Planet Earth (MPE) 2013. A major theme included in the MPE initiative is public outreach (e.g. in the form of a mathematics exhibition) to increase awareness of the importance and essential...
nature of mathematics in processes affecting the Planet Earth. Such a major undertaking will need inputs from many different organizations and individuals, including professional mathematicians, designers, NGOs, museums, college and school teachers and students. The successful candidate will be involved in the coordination of this whole activity, which also involves preparatory activities such as soliciting and sorting proposals for mathematical exhibition modules.

Qualification and Experience: Masters in mathematics or allied subjects (physics, engineering, computer science etc.); A strong dedication to the main aim of this outreach activity; Dexterity with computers and strong written and oral communication skills. (Experience in Art, Design and Technology is desirable.)

The position is temporary, for one year in the first instance, and extendable by one year in deserving cases. A consolidated honorarium, based on qualifications and experience, will be paid as per norms.

How to Apply: Interested candidates, please send your CV and a detailed statement of interest in and your suitability for this position, by E-mail to mpe2013india@gmail.com (preferably) or by post to

MPE-2013
International Centre for Theoretical Sciences
TIFR Centre Building
IISc campus, Bengaluru 560 012

The first round of selection process will take place towards the end of January 2012.

Any enquiries should be sent to
mpe2013india@gmail.com


Mathematics of Planet Earth 2013

The year 2013 has been marked for various mathematical activities under a wide umbrella of initiatives called the Mathematics of the Planet Earth (MPE) 2013, which will focus on mathematical research in areas of relevance to the various processes that affect the planet earth. The dynamics of the oceans and the atmosphere and the changes in the climate are of course the obvious topics that are very important for the life on planet earth and make use of mathematics in an essential way. In addition to these, a multitude of other topics are of relevance to MPE-2013, including the financial and economic systems, the energy production and utilization, spread of epidemics at the population level, ecology and genomics of species, just to name a few. A comprehensive and ever-evolving list of such topics can be found on the website E-mail: http://www.crm.umontreal.ca/Math2013/en/theme.php

To stimulate imagination on the many domains where mathematics plays a crucial role in planetary issues, the following four (non exhaustive) themes are proposed as part of MPE-2013:

I. A planet to discover: oceans; meteorology and climate; mantle processes, natural resources, celestial mechanics.
II. A planet supporting life: ecology, biodiversity, evolution.
III. A planet organized by humans: political, economic, social and financial systems; organization of transport and communications networks; management of resources; energy.
IV. A planet at risk: climate change, sustainable development, epidemics; invasive species, natural disasters.

Mathematics plays a crucial role in two ways in this research since it is used as a universal language and tool for any quantitative research in all the sciences, including biology, economics, etc. Thus it is an essential component of any multi- or inter-disciplinary research. Furthermore, fundamental mathematical questions arise out of these research topics. The activities during the MPE 2013 will highlight both these aspects.

Many international bodies, including research institutes, societies, public organizations, scientific journals, and teacher associations, are partners in MPE 2013. The International Centre for Theoretical Sciences (ICTS) of the Tata Institute of Fundamental Research (TIFR) is a partner institute in the MPE initiative. The main goals of ICTS are to foster research, be a resource for high level education and training, and reach out to the larger society by being a node for scientific information and values. (More detailed information about the past, current, and proposed activities is available at http://www.icts.res.in/.) The specific role that ICTS would play will be to liaise with the Indian scientific community, possibly in collaboration with researchers around...
the world, in order to conduct workshops, thematic programs, and conferences on topics that are aligned with the MPE initiative.

Another major theme of the MPE initiative will be public outreach to increase awareness of the importance and essential nature of mathematics in tackling these problems, and to bring out the relevance and usefulness of mathematics to a wider section of society than just those who use it professionally. One particular activity that is being actively pursued at the international level is a “Competition for an open source exhibition of virtual modules.” Detailed information about this can be found at http://www.mpe2013.org/competition/.

The outreach activities that we hope to plan in India will be in the form of mathematics exhibition(s), interactive sessions involving mathematical discussions and experiments for children, youth, and teachers, special guest lectures given by renowned mathematicians targeted, primarily, at the non-mathematics community. Such activities will provide the youth and children an opportunity to interact with some well known names in the field of mathematics, and will inspire them to study higher mathematics and its applications. There are also plans for an India-specific call for exhibits (and possibly a competition), in cooperation with the global competition for exhibits.

Some of the faculty at TIFR Centre for Applicable Mathematics, Bangalore are presently involved in taking this ICTS initiative forward. We are actively looking for organizations and individuals interested in taking part in this initiative and would love to hear about possible interests in the scientific as well as outreach activities (E-mail contact: mpe2013@icts.res.in)

Some References:
http://www.mpe2013.org and
http://www.mpe2013.org/competition/

20th Mathematics Training and Talent Search Programme
(Funded by National Board for Higher Mathematics)

M.T. & T. S. 2012, Website: http://www.mtts.org.in/

Aims: The aim of the programme is to expose bright young students to the excitement of doing mathematics, to promote independent mathematical thinking and to prepare them for higher aspects of mathematics.

Academic Programmes: The programme will be at three levels: Level O, Level I and Level II. In Level O there will be courses in Linear Algebra, Analysis and Number Theory/Combinatorics. In Levels I and II there will be courses on Algebra, Analysis and Topology. There will be seminars by students at all Levels.

The faculty will be active mathematicians with a commitment to teaching and from various leading institutions. The aim of the instructions is not to give routine lectures and presentation of theorem-proofs but to stimulate the participants to think and discover mathematical results.

Eligibility:

- **Level O**: Second year undergraduate students with Mathematics as one of their subjects.
- **Level I**: Final year undergraduate students with Mathematics as one of their subjects. Participants of the previous year Level O Programme.
- **Level II**: First year postgraduate students with Mathematics as one of their subjects. Participants of the previous year Level I Programme.

Venues & Duration: MTTS 2012 will be held at three different places:

- Indian Institute of Technology (IIT), Kanpur. Duration is May 21 – June 16, 2012.
- Indian Institute of Technology (IIT), Guwahati. Duration is May 14 – June 09, 2012.
- Bharathidasan University, Tiruchirappalli. Duration is May 14 – June 09, 2012.

How to Apply?: Details and application forms can be had from the Head, Department of Mathematics of your Institution. The application forms can also be downloaded from the mtts website:

http://www.mtts.org.in/

Students can also apply online by logging in to this site from January 14, 2012.

The completed application form should reach the programme director latest by February 23, 2012. Please go
through the MTTS-FAQ at the website before sending any queries.

**Programme Director:**

Prof. S. Kumaresan  
Director, MTTS Programme  
Department of Mathematics and Statistics  
University of Hyderabad  
P.O. Central University  
Hyderabad 500 046  
Tel: (040) 6679 4059 (O)  
E-mail: kumaresa@gmail.com; mttsprogramme@gmail.com

**Selection:** The selection will be purely on merit, based on consistently good academic record and the recommendation letter from a mathematics professor closely acquainted with the candidate.

Only selected candidates will be informed of their selection by the 3rd week of March 2012. The list of selected candidates will be posted on the MTTS site in the 2nd week of March 2012.

Candidates selected for the programme will be paid sleeper class return train fare by the shortest route and will be provided free board and lodging for the duration of the course.

The readers may download the Mathematics Newsletter from the RMS website at  
www.ramanujanmathsociety.org  
www.rmsconfmathau.org