GELFAND'S PROOF OF WIENER'S THEOREM

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1. Introduction

The following theorem was proved by the famous mathematician Norbert Wiener. Wiener's proof can be found in his book [5].

Theorem 1.1. (**Wiener's Theorem**) *Let f be a periodic function on $[-\pi, \pi]$. Suppose f has an absolutely convergent Fourier series and $f(t) \neq 0$ for all $t \in [-\pi, \pi]$. Then $1/f$ also has absolutely convergent Fourier series.*

Gelfand gave a proof of this theorem using the techniques from Banach Algebras in his celebrated paper [1]. This was the first paper in which theory of Banach algebras was developed systematically. Gelfand's proof is much shorter than the original proof of Wiener and hence it attracted the attention of Mathematicians to the theory of Banach algebras. In this article, [1] we present Gelfand's proof. In order to make this article self contained, we also give some basic facts about Banach algebras required to understand Gelfand's proof. These are kept to the bare minimum. Interested reader can consult [3] for a deatailed and thorough treatment of Banach algebras. Standard results from Functional Analysis are assumed. These can be found in any introductory text on Functional Analysis, for example [2] or [4]. Any one who has done one course in Functional Analysis should be able to understand this proof.

Both Wiener as well as Gelfand have been illustrious mathematicians with epochmaking contributions to various fields. Their work has also influenced developments of several branches of mathematics. Interested reader can find the biographical information about these mathematicians in the following web cites.

(1) Biography of Wiener
(2) Biography of Gelfand

2. preliminaries

In this section, we give some basic definitions and examples.

[1]This is an expanded version of a talk given at the Workshop on Fourier Analysis in Department of Mathematics, Indian Institute of Technology Madras, on July 10, 2004.

Definition 2.1. A *complex algebra* A is a ring that is also a complex vector space such that

$$(\alpha a)b = \alpha(ab) = a(\alpha b) \quad \text{for all} \quad a, b \in A$$

A is called *commutative* if $ab = ba$ for all $a, b \in A$.
We shall assume that A has a unit element 1 satisfying $1a = a = a1$ for all $a \in A$.

We need the following concepts.

Definition 2.2. Let A be a complex algebra with unit 1.

- Let $a \in A$. If there exists $b \in A$ such that $ab = 1 = ba$, then a is said to be *invertible* and b is called *inverse* of A. It is easy to prove that such an inverse, if exists, is unique. A is said to be a *division algebra* if every nonzero element in A is invertible.
- A subset I of A is called an *ideal* if it is a subspace as well as a two-sided ideal in the sense of ring theory. This means that if $a, b \in I, \alpha, \beta \in \mathbb{C}, c \in A$, then $\alpha a + \beta b, ac, ca \in I$. If I is an ideal and $x \in A$, then the set

$$x + I := \{x + a : a \in I\}$$

  is called a *coset*. The set of all such cosets is denoted by A/I. There is a natural way of defining algebraic operarions on A/I in such a way that it becomes a complex algebra, known as *quotient algebra*.
- An ideal is called a *proper ideal* if it is different from A. A *maximal ideal* is a proper ideal not properly contained in any proper ideal. Thus M is a maximal ideal if and only if $M \neq A$ and whenever $M \subseteq I \subseteq A$ and I is an ideal , we have $I = M$ or $I = A$. It can be proved easily that M is a maximal ideal if and only if A/M is a division algebra.
- A *multiplicative linear functional* on A is a function $\phi : A \to \mathbb{C}$ satisfying the following:
  $\phi(\alpha a + \beta b) = \alpha\phi(a) + \beta\phi(b)$ for all $a, b \in A$ and $\alpha, \beta \in \mathbb{C}$
  $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in A$.
  It can be proved easily that if ϕ is a nonzero multiplicative linear functional, then the null space $N(\phi) := \{a \in A : \phi(a) = 0\}$ is a maximal ideal in A.

Definition 2.3. Banach algebras Let A be a complex algebra. An *algebra norm on* A is a function $\|.\| :\to \mathbb{R}$ satisfying:

(1) $\|a\| \geq 0$ for all $a \in A$ and $\|a\| = 0$ if and only if $a = 0$.
(2) $\|\alpha a\| = |\alpha|\|a\|$ for all $a \in A$ and $\alpha \in \mathbb{R}$
(3) $\|a + b\| \leq \|a\| + \|b\|$ for all $a, b \in A$.
(4) $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in A$.

A *complex normed algebra* is a complex algebra A with an algebra norm defined on it. A *Banach algebra* is a complete normed algebra.

We shall assume that A is *unital*, that is A has unit 1 with $\|1\| = 1$.

Next we give two examples of Banach algebras. Many more examples can be found in [3].

Example 2.4. Let X be a compact Hausdorff space, and let $C(X)$ denote the set of all complex valued continuous functions. Then $C(X)$ is a commutative Banach algebra under pointwise operations and the sup norm given by

$$\|f\| := \sup\{|f(x)| : x \in X\}, \quad f \in C(X)$$

Example 2.5. Wiener Algebra Let W be the set of all complex valued functions on $[-\pi, \pi]$ with absolutely convergent Fourier series, that is, functions of the form

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \exp(int), \quad , t \in [-\pi, \pi]$$

with $\|f\| := \sum_n |c_n| < \infty$. W is a complex Banach algebra.

Thus to prove Wiener's theorem we need to prove the following: If $f \in W$ and $f(t) \neq 0$ for all $t \in [-\pi, \pi]$, then f is invertible.

3. Proofs

Theorem 3.1. *Let A be a complex Banach algebra with unit 1. If $a \in A$ and $\lambda \in \mathbb{C}$ with $\|a\| < |\lambda|$, then $\lambda - a$ is invertible and*

$$\|(\lambda - a)^{-1}\| \leq \frac{1}{|\lambda| - \|a\|}$$

Proof. Consider the infinite series $\sum_{n=0}^{\infty} \dfrac{a^n}{\lambda^{n+1}}$. Since $\|a\|/|\lambda| < 1$, this is an absolutely convergent series. Hence by completeness of A, it converges. Let b denote the sum and $b_n := \frac{1}{\lambda} + \frac{a}{\lambda^2} + \ldots + \frac{a^n}{\lambda^{n+1}}$ denote the partial sum of the series. Then for each n, $(\lambda - a)b_n = 1 - (a/\lambda)^{n+1}$. Taking limits as $n \to \infty$, we get $(\lambda - a)b = 1$. Similarly, $b(\lambda - a)b = 1$. This shows that $\lambda - a$ is invertible. Next,

$$\|(\lambda - a)^{-1}\| = \|b\| \leq \sum_{n=0}^{\infty} \|\frac{a^n}{\lambda^{n+1}}\| \leq \sum_{n=0}^{\infty} \frac{\|a\|^n}{|\lambda|^{n+1}} \leq \frac{1}{|\lambda| - \|a\|}$$

□

Corollary 3.2. *Suppose $a \in A$ is invertible and $b \in A$ is such that $\|a - b\| < \dfrac{1}{\|a^{-1}\|}$. Then b is also invertible and*

$$\|a^{-1} - b^{-1}\| \leq \frac{\|a^{-1}\|^2 \|a - b\|}{1 - \|a - b\| \|a^{-1}\|}$$

This means that the set of all invertible elements in A is an open set in A and the map $a \to a^{-1}$ defined on this set is continuous.

Proof. Since $\|1 - a^{-1}b\| = \|a^{-1}a - a^{-1}b\| \leq \|a^{-1}\|\|a - b\| < 1$, by the above Theorem 3.1, $a^{-1}b$ and hence b is invertible. Further, note that $\|a^{-1} - b^{-1}\| = \|b^{-1}(a - b)a^{-1}\| \leq \|b^{-1}\|\|(a - b)\|\|a^{-1}\|$. (*)

Since
$\|b^{-1}\| \leq \|b^{-1} - a^{-1}\| + \|a^{-1}\|$, we obtain from (*) above,
$\|b^{-1}\| \leq \|b^{-1}\|\|(a - b)\|\|a^{-1}\| + \|a^{-1}\|$, that is,
$\|b^{-1}\| \leq \dfrac{\|a^{-1}\|}{1 - \|(a - b)\|\|a^{-1}\|}$.
Using this estimate (*) becomes

$$\|a^{-1} - b^{-1}\| \leq \frac{\|a^{-1}\|^2\|a - b\|}{1 - \|a - b\|\|a^{-1}\|}$$

□

Corollary 3.3. *Let ϕ be a nonzero multiplicative linear functional on A. Then $\|\phi\| = 1$.*

Proof. Since ϕ is multiplicative, $(\phi(1))^2 = \phi(1^2) = \phi(1)$ and since it is also nonzero, $\phi(1) = 1$. This also means that if $a \in A$ is invertible, then $\phi(a) \neq 0$. Now
$\|\phi\| := \sup\{\phi(x) : x \in A, \|x\| \leq 1\} \geq \phi(1) = 1$.

Next let $x \in A$ and $\|x\| \leq 1$. Then for every $\lambda \in \mathbb{C}$ with $|\lambda| > 1$, by the above Theorem 3.1, $\lambda - x$ is invertible. Hence $0 \neq \phi(\lambda - x) = \lambda - \phi(x)$. Thus $|\phi(x)| \leq 1$. This implies that $\|\phi\| \leq 1$ and completes the proof. □

Theorem 3.4. *(***Gelfand Mazur Theorem***)*
Every complex Banach division algebra is isometrically isomorphic to $\mathbb{C}$.

Proof. Let A be a complex Banach division algebra. Let $a \in A$. We shall prove $a = z$ (that is $z1$) for some $z \in \mathbb{C}$. Suppose $a - z \neq 0$ for all $z \in \mathbb{C}$. Then, since A is a division algebra, $a - z$ is invertible for all $z \in \mathbb{C}$. In particular, a is invertible. By (a corollary of) Hahn-Banach theorem, there exists a continiuos linear functional ϕ on A such that $\phi(a^{-1}) \neq 0$. Now define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) := \phi((a - z)^{-1}), \quad z \in \mathbb{C}$.
Now let $z \in \mathbb{C}$ and consider
$\dfrac{f(z + h) - f(z)}{h} = \dfrac{\phi((a - (z + h))^{-1} - \phi((a - z)^{-1}))}{h}$
$= \phi(\dfrac{(a - (z + h))^{-1} - (a - z)^{-1}}{h}) = -\phi((a - z)^{-1})(a - (z + h))^{-1})$
Now using continuity of ϕ and of the map $b \to b^{-1}$, (see Corollary 3.2) it can be shown that for every $z \in \mathbb{C}$, $\lim_{h \to 0} \dfrac{f(z + h) - f(z)}{h}$ exists and, in fact, equals $-\phi((a - z)^{-2})$. This shows that f is an entire function. Also, for $|z| > \|a\|$, $|f(z)| \leq \|\phi\|\|(a - z)^{-1}\| \leq \dfrac{1}{|z| - \|a\|}$ by Theorem

3.1. Hence, $|f(z)| \to 0$ as $|z| \to \infty$. This, in particular, implies that f is bounded and hence by Louville's theorem, f is constant. Moreover, since $|f(z)| \to 0$ as $|z| \to \infty$, this constant must be 0. In particular, $f(0) := \phi(a^{-1}) = 0$, a contradiction.

□

If I is a closed ideal in a normed algebra A, then the quotient algebra A/I can be made into a normed algebra by defining

$$\|x + I\| := \inf\{\|x + y\| : y \in I\}, \quad x \in A$$

Further, if A is a Banach algebra, then A/I is also a Banach algebra. Also, if M is a maximal ideal, then M is a closed ideal and A/M is a Banach division algebra. This is used in the next corollary.

Corollary 3.5. *Let A be a complex commutative Banach algebra with unit 1. If $a \in A$ is not invertible, then there exists a nonzero multiplicative linear functional ϕ such that $\phi(a) = 0$.*

Proof. Consider $I = aA := \{ab : b \in A\}$. Since a is not invertible, I is a proper ideal in A and is hence contained in a maximal ideal M. Then A/M is a Banach division algebra and is hence isomorphic to $\mathbb{C}$. Let this isomorphism be ψ and define $\phi : A \to \mathbb{C}$ by $\phi(x) := \psi(x + M), \quad x \in A$. Then ϕ is the required multiplicative linear functional. □

Proposition 3.6. *Let ϕ be a nonzero multiplicative linear functional on the Wiener algebra W. Then there exists $t_0 \in [-\pi, \pi]$ such that $\phi(f) = f(t_0)$ for all $f \in W$*

Proof. Consider the function g defined by $g(t) = \exp(it), \quad t \in [-\pi, \pi]$. Then $g \in W$ and $\|g\| = 1$. Let $\lambda = \phi(g)$. Then $|\lambda| = |\phi(g)| \leq \|\phi\|\|g\| = 1$. (Note $\|\phi\| = 1$ by Corollary 3.3.) Further g is invertible and $g^{-1}(t) = \exp(-it), \quad t \in [-\pi, \pi]$. Thus $\|g^{-1}\| = 1$. Also, since ϕ is a nonzero multiplicative linear functional, $\phi(1) = 1$ and hence $\phi(g^{-1}) = 1/\lambda$. This implies that $|1/\lambda| \leq 1$. Hence $|\lambda| = 1$. Therefore there exists $t_0 \in [-\pi, \pi]$ such that $\lambda = \exp(it_0)$. Now let $f \in W$. Then

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \exp(int), \quad , t \in [-\pi, \pi]$$

Thus $f = \sum_{n=-\infty}^{\infty} c_n g^n$. Hence

$$\phi(f) = \sum_{n=-\infty}^{\infty} c_n \phi(g^n) = \sum_{n=-\infty}^{\infty} c_n (\phi(g))^n = \sum_{n=-\infty}^{\infty} c_n \lambda^n = \sum_{n=-\infty}^{\infty} c_n \exp(int_0) = f(t_0)$$

□

Now we have all tools required to prove Wiener's theorem. Recall the observation at the end of the last section that to prove Wiener's theorem we need to prove the following: If $f \in W$ and $f(t) \neq 0$ for all

$t \in [-\pi, \pi]$, then f is invertible.
Suppose such f is not invertible. Then by Corollary 3.5, there exists a nonzero multiplicative linear functional ϕ on W such that $\phi(f) = 0$. Next, by Proposition 3.6, there exists $t_0 \in [-\pi, \pi]$ such that $\phi(f) = f(t_0)$. Thus $f(t_0) = 0$, a contradiction. This is the essence of Gelfand's proof of Wiener's theorem.

References

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