Numerical Methods for Partial Differential Equations using Radial Basis Functions

Chandhini.G
Department of Mathematics

chandhini@iitm.ac.in
OUTLINE

• Gridfree methods

• Radial Basis Function (RBF) Interpolation

• Collocation using RBFs

• Preconditioning

• Finite difference type formulation using RBFs
Gridfree methods for solving PDEs

• Gridfree methods and their importance
• Multivariate Interpolation (scattered data) in $\mathbb{R}^d$, $d > 1$

$$s(\bar{x}) = \sum_{j=1}^{n} \lambda_j B_j(\bar{x})$$

– Different methods - their drawbacks

– Radial Basis Functions
In one dimension, data can be interpolated smoothly using splines.

 Doesn't matter if nodes are scattered rather than equispaced.
On a regular 2-D grid, data can be interpolated smoothly by splines.

On scattered nodes, smooth splines are difficult.
A radial basis function interpolant can be smooth and accurate on any set of nodes in any dimension.
RBF - As an Interpolation Tool

Definition  Let $X \subseteq \mathbb{R}^d$, $d \geq 1$ be a normed linear space. A function $\psi : X \rightarrow \mathbb{R}$ is said to be radial if there exists a univariate function $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}$ such that $\psi(\bar{x}) = \phi(\|\bar{x}\|)$ for all $\bar{x} \in X$. A radial basis function is any translates of $\phi$, i.e, a function of the form $\psi(\bar{x} - \bar{\xi}) = \phi(\|\bar{x} - \bar{\xi}\|)$, where $\bar{\xi}$ is any prescribed point of $X$.

$\|\cdot\|$, some norm in $X \subseteq \mathbb{R}^d$ - usually Euclidean norm.
Examples: \( (\phi(r), r \geq 0) \)

1. Infinitely Smooth Radial Functions (feature a shape parameter \( c \neq 0 \))

   Gaussian ( \( e^{-(r/c)^2} \) )

   Multiquadric ( \( \sqrt{c^2 + r^2} \) )

   Inverse Multiquadric ( \( \frac{1}{\sqrt{c^2 + r^2}} \) )
Examples: \((\phi(r), r \geq 0)\)

1. Infinitely Smooth Radial Functions (feature a shape parameter \(c \neq 0\))

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   Inverse Multiquadric \((\frac{1}{\sqrt{c^2+r^2}})\)

2. Piecewise Smooth Radial Functions

   Linear \((r)\)

   Cubic \((r^3)\)

   Thin plate Splines \((r^2\log(r))\)
Definition  Given a set of $n$ distinct data $(\bar{x}_i, f_i)_{i=1}^n$, where $\bar{x}_i \in \mathbb{R}^d$, $d \geq 1$, the RBF interpolant is given by,

$$s(x) = \sum_{j=1}^n \lambda_j \phi(\| \bar{x} - \bar{x}_j \|)$$  \hspace{1cm} (1)

Applying interpolation condition,

$$s(x_i) = f_i, \; i = 1, \ldots n$$  \hspace{1cm} (2)

leads to the following linear system,  \hspace{1cm} $A\lambda = \bar{f}$

where $A_{i,j} = \phi(\| \bar{x}_i - \bar{x}_j \|)$
Uniqueness - nonsingularity of $A$

**Definition** A real-valued function $\phi$ is **positive definite** on $\mathbb{R}^d$ if and only if it satisfies

$$\sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_j \alpha_k \phi(\bar{x}_j - \bar{x}_k) > 0$$

for any $n$ distinct points $\bar{x}_1, \ldots, \bar{x}_n \in \mathbb{R}^d$ and $\alpha = [\alpha_1, \ldots, \alpha_n]^T \neq 0$
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**Definition** A function $\phi$ is said to be **completely monotone** on $[0, \infty)$ if,

1. $\phi \in C[0, \infty) \cap C^\infty(0, \infty)$

2. $(-1)^k \phi^{(k)}(r) \geq 0, \ r > 0, \ k = 0, 1, 2, \ldots$
Theorem (Schoenberg(1938)):

If $\phi : [0, \infty) \to \mathbb{R}$ is a non-constant completely monotone function, then $\phi(\|\cdot\|^2)$ is positive definite and radial on $\mathbb{R}^d$, $d \geq 1$.

Examples: Gaussian, Inverse MQ (But not MQ or TPS).
Theorem (Schoenberg(1938)):

If \( \phi : [0, \infty) \rightarrow \mathbb{R} \) is a non-constant completely monotone function, then \( \phi(\| \cdot \|_2^2) \) is positive definite and radial on \( \mathbb{R}^d, \ d \geq 1 \).

Examples: Gaussian, Inverse MQ (But not MQ or TPS).

**Augmented RBF Method**

Interpolant is of the form,

\[
s(x) = \sum_{j=1}^{n} \lambda_j \phi(\| \bar{x} - \bar{x}_j \|) + \sum_{k=1}^{M} \gamma_k p_k(\bar{x}), \ x \in \mathbb{R}^d
\]

(3)

where \( p_1, \ldots, p_M \) form a basis for the space \( \prod_{m-1}^{d} \) (polynomials of degree \( \leq m - 1 \) in \( d \) variables).
Definition A real valued continuous function $\phi$ is called **conditionally positive definite** of order $m$ on $\mathbb{R}^d$, if,

$$\sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_j \alpha_k \phi(x_j - x_k) > 0$$
Definition  A real valued continuous function $\phi$ is called **conditionally positive definite** of order $m$ on $\mathbb{R}^d$, if,

$$\sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_j \alpha_k \phi(\bar{x}_j - \bar{x}_k) > 0$$

for any $\bar{x}_1, \ldots, \bar{x}_n \in \mathbb{R}^d$ and $\alpha = [\alpha_1, \ldots, \alpha_n]^T \neq 0$ satisfying

$$\sum_{j=1}^{n} \alpha_j \bar{x}_j^\nu = 0, \quad |\nu| = \sum_{i=1}^{d} \nu_i < m, \quad \bar{x}^\nu = x_1^{\nu_1} x_2^{\nu_2} \ldots x_n^{\nu_n}$$
Definition A real valued continuous function \( \phi \) is called conditionally positive definite of order \( m \) on \( \mathbb{R}^d \), if,

\[
\sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_j \alpha_k \phi(\bar{x}_j - \bar{x}_k) > 0
\]

for any \( \bar{x}_1, \ldots, \bar{x}_n \in \mathbb{R}^d \) and \( \alpha = [\alpha_1, \ldots, \alpha_n]^T \neq 0 \) satisfying

\[
\sum_{j=1}^{n} \alpha_j \bar{x}_j^\nu = 0, \quad |\nu| = \sum_{i=1}^{d} \nu_i < m, \quad \bar{x}^\nu = x_1^{\nu_1}x_2^{\nu_2} \ldots x_n^{\nu_n}
\]

Examples: MQ - conditionally p.d of order 1

TPS - conditionally p.d of order 2
To accommodate extra degrees of freedom, enforce,

\[
\sum_{j=1}^{n} \lambda_j p_l(\bar{x}_j) = 0, \ l = 1, \ldots, M \tag{4}
\]

leads to,

\[
\begin{pmatrix}
A & P \\
P^T & 0
\end{pmatrix}
\begin{pmatrix}
\lambda \\
\gamma
\end{pmatrix}
=
\begin{pmatrix}
f \\
0
\end{pmatrix} \tag{5}
\]
To accommodate extra degrees of freedom, enforce,

$$\sum_{j=1}^{n} \lambda_j p_l(\bar{x}_j) = 0, \ l = 1, \ldots, M$$

leads to,

$$\begin{pmatrix} A & P \\ P^T & 0 \end{pmatrix} \begin{pmatrix} \lambda \\ \gamma \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

**Theorem** (Michelli-1986) The real valued function $\phi$ is conditionally p.d of order $m$ on $\mathbb{R}^d$ and the points $\{\bar{x}_1, \ldots, \bar{x}_n\}$ are such that $\text{Rank}(P) = M$, then the linear system in the augmented RBF interpolation is nonsingular.
Accuracy of the Interpolant and Stability of the System

Infinitely Smooth RBFs:

1. depends on number of data and the shape parameter $c$
   - can improve the accuracy by increasing $c$ significantly
   - but stability is lost by increasing the number of points or by increasing the value of $c$ due to the exponential increase in the condition number of the interpolation matrix

2. for a fixed $c$, converges exponentially ($O(e^{-\frac{const}{h}})$)
Accuracy of the Interpolant and Stability of the System

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2. for a fixed $c$, converges exponentially ($O\left(e^{-\frac{\text{const}}{h}}\right)$)

Piecewise Smooth RBFs:
1. convergence depends on the smoothness and the space dimension
2. Stability get affected by increase in number of points and also by smoothness
Given data \( \{\bar{x}_i, \mathcal{L}_i f\}, \ i = 1, \ldots, n \), where \( \mathcal{L} = \{\mathcal{L}_1, \ldots, \mathcal{L}_n\} \), are linear differential operators.

\[
s(\bar{x}) = \sum_{j=1}^{n} \lambda_j \mathcal{L}_j^2 \phi(\|\bar{x} - \bar{x}_j\|), \quad \bar{x} \in \mathbb{R}^d
\]  

(6)
Hermite RBF Interpolation - Z.Wu (1992)

Given data \( \{ \bar{x}_i, \mathcal{L}_i f \} \), \( i = 1, \ldots, n \), where \( \mathcal{L} = \{ \mathcal{L}_1, \ldots, \mathcal{L}_n \} \), are linear differential operators.

\[
s(\vec{x}) = \sum_{j=1}^{n} \lambda_j \mathcal{L}_j^2 \phi(\|\vec{x} - \bar{x}_j\|), \quad \vec{x} \in \mathbb{R}^d
\]  \hspace{1cm} (6)

Apply condition,

\[
\mathcal{L}_i s = \mathcal{L}_i f, \quad i = 1, \ldots, n
\]  \hspace{1cm} (7)

Leads to \( A\lambda = \mathcal{L} f \), where \( A_{ij} = \mathcal{L}_i^1 \mathcal{L}_j^2 \phi \)
Hermite RBF Interpolation - Z.Wu (1992)

Given data \( \{\bar{x}_i, L_i f\} \), \( i = 1, \ldots, n \), where \( L = \{L_1, \ldots, L_n\} \), are linear differential operators.

\[
s(\bar{x}) = \sum_{j=1}^{n} \lambda_j L_j^2 \phi(\|\bar{x} - \bar{x}_j\|), \quad \bar{x} \in \mathbb{R}^d
\]  

(6)

Apply condition, \( L_i s = L_i f, \quad i = 1, \ldots, n \)  

(7)

Leads to \( A \lambda = L f \), where \( A_{ij} = L_i^1 L_j^2 \phi \)

- \( A \) is non-singular for the class of radial functions defined previously.
RBF for solving PDEs
RBF for solving PDEs

• Collocation Methods
RBF for solving PDEs

- Collocation Methods
- Variational Formulations
RBF for solving PDEs

- Collocation Methods
- Variational Formulations
- Boundary Element Methods (Method of fundamental solutions)
Collocation Methods

Consider the boundary value problem,

\[
\begin{align*}
  \mathcal{L}[u](\bar{x}) &= f(\bar{x}), \quad \bar{x} \in \Omega, \\
  \mathcal{B}[u](\bar{x}) &= g(\bar{x}), \quad \bar{x} \in \partial \Omega,
\end{align*}
\]

where \( \mathcal{L} \) and \( \mathcal{B} \) linear partial differential operators in the domain \( \Omega \) and \( \partial \Omega \) respectively.

Given \( (\bar{x}_j)_{j=1}^n \), \( n_B \) - boundary nodes

\[ n - n_B \] - interior nodes.
Asymmetric Collocation - Kansa (1990)

\[ u(\bar{x}) = \sum_{j=1}^{n} \lambda_j \phi(\|\bar{x} - \bar{x}_j\|) \]  \hspace{1cm} (9)

Apply the operators \( \mathcal{L} \) and \( \mathcal{B} \) on \( u \) for interior and boundary points respectively,

\[
\begin{pmatrix}
\mathcal{B}_1 \phi \\
\mathcal{L}_1 \phi
\end{pmatrix}
\lambda =
\begin{pmatrix}
g \\
f
\end{pmatrix}
\]  \hspace{1cm} (10)
Asymmetric Collocation - Kansa(1990)

\[ u(\bar{x}) = \sum_{j=1}^{n} \lambda_j \phi(\|\bar{x} - \bar{x}_j\|) \]  

Apply the operators \( \mathcal{L} \) and \( \mathcal{B} \) on \( u \) for interior and boundary points respectively,

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\begin{pmatrix}
\mathcal{B}_1\phi \\
\mathcal{L}_1\phi
\end{pmatrix}
\lambda = 
\begin{pmatrix}
g \\
f
\end{pmatrix}
\]  

– Easy to implement
– linear system is non-symmetric
– no theory has been developed

\[ u(\bar{x}) = \sum_{j=1}^{n_B} \lambda_j B_2 \phi(\|\bar{x} - \bar{x}_j\|) + \sum_{j=n_B+1}^{n} \lambda_j L_2 \phi(\|\bar{x} - \bar{x}_j\|) + P_m(\bar{x}) \quad (11) \]

Application of the operators \( \mathcal{L} \) and \( B \) on \( u \) for interior and boundary points respectively, and along with the orthogonality condition, leads to,

\[
\begin{pmatrix}
B_1 B_2 \phi & B_1 L_2 \phi & B_1 P_m \\
L_1 B_2 \phi & L_1 L_2 \phi & L_1 P_m \\
B_1 P_m^T & L_1 P_m^T & 0
\end{pmatrix}
\begin{pmatrix}
\lambda_B \\
\lambda_L \\
\gamma
\end{pmatrix}
= \begin{pmatrix}
g \\
f \\
0
\end{pmatrix} \quad (12)
\]
• Symmetric Collocation - G.E.Fasshauer (1997)

\[
u(\bar{x}) = \sum_{j=1}^{n_B} \lambda_j B_2 \phi(\|\bar{x} - \bar{x}_j\|) + \sum_{j=n_B+1}^{n} \lambda_j L_2 \phi(\|\bar{x} - \bar{x}_j\|) + P_m(\bar{x}) \quad (11)
\]

Application of the operators \(L\) and \(B\) on \(u\) for interior and boundary points respectively, and along with the orthogonality condition, leads to,

\[
\begin{pmatrix}
B_1 B_2 \phi & B_1 L_2 \phi & B_1 P_m \\
L_1 B_2 \phi & L_1 L_2 \phi & L_1 P_m \\
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\lambda_B \\
\lambda_L \\
\gamma
\end{pmatrix}
= 
\begin{pmatrix}
g \\
f \\
0
\end{pmatrix}
\quad (12)
\]

– Linear system is symmetric and non-singular
Seminar-1


\[ u(\bar{x}) = \sum_{j=1}^{n_B} \lambda_j B_2 \phi(\|\bar{x} - \bar{x}_j\|) + \sum_{j=n_B+1}^{n} \lambda_j L_2 \phi(\|\bar{x} - \bar{x}_j\|) + P_m(\bar{x}) \]  \hspace{1cm} (11)

Application of the operators \( L \) and \( B \) on \( u \) for interior and boundary points respectively, and along with the orthogonality condition, leads to,

\[
\begin{pmatrix}
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    L_1 B_2 \phi & L_1 L_2 \phi & L_1 P_m \\
    B_1 P^T_m & L_1 P^T_m & 0
\end{pmatrix}
\begin{pmatrix}
    \lambda_B \\
    \lambda_L \\
    \gamma
\end{pmatrix}
=
\begin{pmatrix}
    g \\
    f \\
    0
\end{pmatrix}
\]  \hspace{1cm} (12)

- Linear system is symmetric and non-singular
- Convergence theory has been developed.
Numerical Examples

Comparison of Multiquadric with finite difference for Steady Convection Diffusion Equation

\[-\nabla^2 u + \bar{b} \cdot \nabla u = f\]

\[BC: c_1 \frac{\partial u}{\partial n} + c_2 u = g\]  \hspace{1cm} (13)
Example 1:
For 2-D convection diffusion equation, with $\Omega = (0, 1)^2$ and $\bar{b} = (b_1, b_2)$,
where,

$$b_1 = -16(x - x^2)^2(y - y^2)(1 - 2y)$$
$$b_2 = 16(x - x^2)(1 - 2x)(y - y^2)^2$$

Solution : $u(x, y) = 16[(6x^2 - 6x + 1)(y - y^2) + (x - x^2)(6y^2 - 6y + 1)]$
Analytical solution

Comparison of r.m.s errors

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Example 2:

For, 2-D convection diffusion equation,

\[ \bar{b} = (Re, 0), \quad f(x, y) = 0, \quad 0 < x, y < 1, \]

BCs: \[ u(x, 0) = 0, \quad u(x, 1) = 0 \]

\[ u(0, y) = \sin \pi y, \quad u(1, y) = 2 \sin \pi y \]
CDS(20X20) solution  MQ solution with 258 centers
20x20 uniform grid

Distribution of 258 centers
Comparison of r.m.s error

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Matrix obtained from RBF interpolation and collocation is, in general, dense and severely ill-conditioned.

Remedy is pre-conditioning(change of basis) and domain decomposition.
Preconditioning

Given system, \[ A\lambda = f \]

Preconditioned system, \[ WA\lambda = Wf \]
Preconditioning

Given system, \( A\lambda = f \)

Preconditioned system, \( WA\lambda = Wf \)

**Approximate Cardinal Basis Function (ACBF) Preconditioner**

- Proposed by Beatson, Cherrie and Mouat
Preconditioning

Given system, \[ A\lambda = f \]

Preconditioned system, \[ WA\lambda = Wf \]

**Approximate Cardinal Basis Function (ACBF) Preconditioner**

- Proposed by Beatson, Cherrie and Mouat
- Idea is to represent the interpolant in terms of cardinal function.
For each center $\bar{x}_i, \ i = 1, \ldots, n$ cardinal function of the interpolation problem,

$$
\psi_i(\bar{x}) = \sum_{j=1}^{n} \omega_{ij} \phi(\|\bar{x} - \bar{x}_j\|), \ i = 1, \ldots, n
$$

(14)

where $\psi_i(\bar{x}_j) = \delta_{ij}$ for $j = 1, \ldots, n$

$\Rightarrow W = (w_{ij})_{i,j=1}^{n}$ is the inverse of the interpolation matrix.
For each center $\bar{x}_i, \ i = 1, \ldots, n$ cardinal function of the interpolation problem,

$$
\psi_i(\bar{x}) = \sum_{j=1}^{n} \omega_{ij} \phi(||\bar{x} - \bar{x}_j||), \ i = 1, \ldots, n 
$$

(14)

where $\psi_i(\bar{x}_j) = \delta_{ij}$ for $j = 1, \ldots, n$

$\Rightarrow W = (w_{ij})_{i,j=1}^{n}$ is the inverse of the interpolation matrix.

- Converting to Cardinal basis function is impractical, hence go for approximate cardinal basis function.
Different strategies for ACBF

- Solving purely local problems. (Minimal-ACBF)
- Solving least-square problems. (LS-ACBF)
Different strategies for ACBF

- Solving purely local problems. (Minimal-ACBF)

- Solving least-square problems. (LS-ACBF)

**Minimal-ACBF**

For each $\bar{x}_i$, let $S_i = \{s_i(1), \ldots, s_i(m)\}$, subset of $\{1, 2, \ldots, n\}$ where $m \ll n$. 
Enforcing cardinal condition only on the subset,

\[ B_i^T \omega_i = e_i \]  \hspace{1cm} (15)

\[ B_i = \begin{pmatrix} 
A_{s_i(1),s_i(1)} & A_{s_i(1),s_i(2)} & \cdots & A_{s_i(1),s_i(m)} \\
A_{s_i(2),s_i(1)} & A_{s_i(2),s_i(2)} & \cdots & A_{s_i(2),s_i(m)} \\
\vdots & \vdots & \ddots & \vdots \\
A_{s_i(m),s_i(1)} & A_{s_i(m),s_i(2)} & \cdots & A_{s_i(m),s_i(1)} 
\end{pmatrix}_{m \times m} \] \hspace{1cm} (16)

\[ W_{ij} = \begin{cases} 
 w_{ik} & \text{if } j = s_i(k) \text{ for } j = 1, 2, \ldots, m \\
0, & \text{otherwise}
\end{cases} \] \hspace{1cm} (17)
LS-ACBF

- Similar to minimal-ACBF, except that cardinal condition is enforced on the whole set \( \{1, 2, \ldots, n\} \) of points

- Leads to an over-determined system \((n \times m)\)

*Including some widely scattered points (special points) gives a better approximation to the cardinal function*
Example

Franke function

Let \( \bar{x} = (\xi, \eta) \in \mathbb{R}^2 \)

\[
f(\xi, \eta) = \frac{3}{4}e^{-\left(\frac{(9\xi-2)^2+(9\eta-2)^2}{4}\right)} + \frac{3}{4}e^{-\left(\frac{(9\xi+1)^2}{49} - \frac{(9\eta-2)^2}{4}\right)10} \\
+ \frac{1}{2}e^{-\left(\frac{(9\xi-7)^2+(9\eta-3)^2}{4}\right)} - \frac{1}{5}e^{-\left(9\xi-4\right)^2-(9\eta-7)^2}
\]
## Comparison table for condition number and iteration counts

### Multi-Quadric:

<table>
<thead>
<tr>
<th>No of centers</th>
<th>condition no:</th>
<th>iter. count</th>
</tr>
</thead>
<tbody>
<tr>
<td>289 Unpreconditioned</td>
<td>7.8 (8)</td>
<td>289</td>
</tr>
<tr>
<td>LS-ACBF</td>
<td>6.7 (2)</td>
<td>75</td>
</tr>
<tr>
<td>LS-ACBF(with spl. points)</td>
<td>2.0 (2)</td>
<td>46</td>
</tr>
<tr>
<td>625 Unpreconditioned</td>
<td>2.9 (9)</td>
<td>593</td>
</tr>
<tr>
<td>LS-ACBF</td>
<td>4.1 (3)</td>
<td>80</td>
</tr>
<tr>
<td>LS-ACBF(with spl. points)</td>
<td>5.4 (2)</td>
<td>47</td>
</tr>
<tr>
<td>1089 Unpreconditioned</td>
<td>4.4 (10)</td>
<td>1089</td>
</tr>
<tr>
<td>LS-ACBF</td>
<td>3.4 (4)</td>
<td>133</td>
</tr>
<tr>
<td>LS-ACBF(with spl. points)</td>
<td>9.8 (2)</td>
<td>76</td>
</tr>
</tbody>
</table>
### Thin-Plate Splines:

<table>
<thead>
<tr>
<th>No of centers</th>
<th>Condition no:</th>
<th>iter. count</th>
</tr>
</thead>
<tbody>
<tr>
<td>289 Unpreconditioned</td>
<td>3.21 (7)</td>
<td>204</td>
</tr>
<tr>
<td>LS-ACBF</td>
<td>2.9 (3)</td>
<td>69</td>
</tr>
<tr>
<td>LS-ACBF(with spl. points)</td>
<td>9.4</td>
<td>31</td>
</tr>
<tr>
<td>625 Unpreconditioned</td>
<td>1.1 (8)</td>
<td>346</td>
</tr>
<tr>
<td>LS-ACBF</td>
<td>2.4 (4)</td>
<td>68</td>
</tr>
<tr>
<td>LS-ACBF(with spl. points)</td>
<td>1.7 (2)</td>
<td>36</td>
</tr>
<tr>
<td>1089 Unpreconditioned</td>
<td>1.9 (8)</td>
<td>454</td>
</tr>
<tr>
<td>LS-ACBF</td>
<td>5.6 (4)</td>
<td>96</td>
</tr>
<tr>
<td>LS-ACBF(with spl. points)</td>
<td>2.6 (2)</td>
<td>52</td>
</tr>
</tbody>
</table>
Finite Difference type formulae based on RBFs

(G.Wright & B.Fornberg)
Finite Difference type formulae based on RBFs

(G.Wright & B.Fornberg)

$L$ - a linear differential operator & $n_i$ - number of nodes in the nbd of $\bar{x}_i$
Finite Difference type formulae based on RBFs

(G.Wright & B.Fornberg)

\( \mathcal{L} \) - a linear differential operator & \( n_i \) - number of nodes in the nbd of \( \bar{x}_i \)

**RBF-Finite difference Formulation**

- Standard RBF Interpolant in Lagrange form

\[
 s(\bar{x}) = \sum_{i=1}^{n_i} \psi_i(\bar{x}) u(\bar{x}_i) \tag{18}
\]

where,

\[
 \psi_i(\bar{x}_k) = \delta_{ik}, \quad \text{for } k = 1, \ldots, n_i \tag{19}
\]
• In terms of $\phi(\|\bar{x} - \bar{x}_j\|)$’s

$$\psi_i(\bar{x}) = \frac{\det(A_i(\bar{x}))}{\det(A)}$$  \hspace{1cm} (20)

where, $A_i(\bar{x})$ is $A$ with the $i^{th}$ row replaced with

$$B(\bar{x}) = [\phi(\|\bar{x} - \bar{x}_1\|) \ \phi(\|\bar{x} - \bar{x}_2\|) \ \ldots \ \phi(\|\bar{x} - \bar{x}_n\|)]$$  \hspace{1cm} (21)
RBF-FD formulae for $\mathcal{L}u(\bar{x}_i)$

Given $n$ points, for $x_i$, consider a stencil of $n_i$ points, to compute $d_j$ such that,

$$\mathcal{L}u(\bar{x}_i) \approx \sum_{j=1}^{n_i} d_j u(\bar{x}_j)$$  \hspace{1cm} (22)
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(22)

Apply the operator $\mathcal{L}$ to (18),

$$\mathcal{L}u(\bar{x}_i) \approx \mathcal{L}s(\bar{x}_i) = \sum_{j=1}^{n_i} \mathcal{L}\psi_j(\bar{x}_i) u(\bar{x}_j)$$

(23)
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$$\Rightarrow \quad d_j = \mathcal{L}\psi_j(\bar{x}_i)$$  \hspace{1cm} (24)

From the definition of $\psi_j$, $d_j$'s can be calculated by solving,

$$Ad = (\mathcal{L}B(\bar{x}_i))^T$$  \hspace{1cm} (25)
RBF-Compact finite difference formulation

Given $u$ at $n$ points and $\mathcal{L}u$ at $l < n$, distinct numbers from $\{1, \ldots, n\}$,

- Hermite-Interpolant in Lagrange form

$$s(\bar{x}) = \sum_{i=1}^{n_i} \psi_i(\bar{x}) u(\bar{x}_i) + \sum_{i=1}^{l} \tilde{\psi}_{\sigma_i}(\bar{x}) \mathcal{L}u(\bar{x}_{\sigma_j})$$  \hspace{1cm} (26)
RBF-Compact finite difference formulation

Given \( u \) at \( n \) points and \( \mathcal{L}u \) at \( l < n \), distinct numbers from \( \{1, \ldots, n\} \),

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\]  

(26)

where,

\[
\psi_i(\bar{x}_k) = \delta_{ik}, \quad \text{for } k = 1, \ldots, n_i
\]  

(27)

\[
\mathcal{L}\psi_i(\bar{x}_k) = 0, \quad k = 1, \ldots, l,
\]  

(28)
and

$$\tilde{\psi}_{\sigma_i}(\bar{x}_k) = 0, \quad k = 1, \ldots, n_i$$ \hspace{1cm} (29)

$$\mathcal{L}\tilde{\psi}_{\sigma_i}(\bar{x}_{\sigma_k}) = \delta_{\sigma_i\sigma_k}, \quad \text{for } k = 1, \ldots, l$$ \hspace{1cm} (30)
and

\[ \tilde{\psi}_{\sigma_i}(\bar{x}_k) = 0, \quad k = 1, \ldots, n_i \]  \hfill (29)

\[ \mathcal{L}\tilde{\psi}_{\sigma_i}(\bar{x}_{\sigma_k}) = \delta_{\sigma_i \sigma_k}, \quad \text{for } k = 1, \ldots, l \]  \hfill (30)

- In terms of \( \phi(\|\bar{x} - \bar{x}_j\|) \)'s and \( \mathcal{L}\phi(\|\bar{x} - \bar{x}_j\|) \)'s

\[ \psi_i(\bar{x}) = \frac{\det(A_i^H(\bar{x}))}{\det(A^H)} \]  \hfill (31)
and

\[
\tilde{\psi}_{\sigma_i}(\bar{x}) = \frac{\det(A_{n+i}^H(\bar{x}))}{\det(A^H)}
\]  

(32)

where \( A_{i}^H(\bar{x}) \) is \( A^H \) with \( i^{th} \) row replaced with

\[
B^H(\bar{x}) = [B(\bar{x}) \mid \mathcal{L}_{2\phi}(\|\bar{x} - \bar{x}_{\sigma_1}\|) \ldots \mathcal{L}_{2\phi}(\|\bar{x} - \bar{x}_{\sigma_l}\|)]
\]  

(33)
RBF-CFD formula for $\mathcal{L}u(\bar{x}_i)$

Given $n$ points, for $x_i$, consider a stencil of $n_i$ points, to compute $d_j$’s and $\tilde{d}_{\sigma_j}$’s such that,

$$\mathcal{L}u(\bar{x}_i) \approx \sum_{j=1}^{n_i} d_j u(\bar{x}_j) + \sum_{j=1}^{l} \tilde{d}_{\sigma_j} \mathcal{L}u(\bar{x}_{\sigma_j})$$  \hspace{1cm} (34)
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Apply the operator $\mathcal{L}$ to (26),

$$\mathcal{L}u(\bar{x}_i) \approx \mathcal{L}s(\bar{x}_i) = \sum_{j=1}^{n_i} \mathcal{L}\psi_j(\bar{x}_i) u(\bar{x}_j) + \sum_{j=1}^{l} \mathcal{L}\tilde{\psi}_{\sigma_j}(\bar{x}_i) \mathcal{L}u(\bar{x}_{\sigma_j})$$  \hspace{1cm} (35)
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$$\Rightarrow d_j = \mathcal{L}\psi_j(\bar{x}_i), \quad \tilde{d}_{\sigma_j} = \mathcal{L}\tilde{\psi}_{\sigma_j}(\bar{x}_i)$$ \hspace{1cm} (36)
RBF-CFD formula for $\mathcal{L}u(\bar{x}_i)$

Given $n$ points, for $x_i$, consider a stencil of $n_i$ points, to compute $d_j$’s and $\tilde{d}_{\sigma_j}$’s such that,

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Apply the operator $\mathcal{L}$ to (26),

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$$\Rightarrow \quad d_j = \mathcal{L}\psi_j(\bar{x}_i), \quad \tilde{d}_{\sigma_j} = \mathcal{L}\tilde{\psi}_{\sigma_j}(\bar{x}_i) \quad (36)$$

which is obtained by solving,

$$A^H [d \mid \tilde{d}]^T = (\mathcal{L}B^H(\bar{x}_i))^T \quad (37)$$
Numerical Examples

1. Poisson Equation in a unit square

\[ \Delta u = f \quad \text{in } \Omega = (0, 1) \times (0, 1), \quad u = g \quad \text{on } \partial \Omega \]

where \( f \) and \( g \) are calculated from the exact solution

\[ u(x, y) = e^{-(x-1/4)^2-(y-1/2)^2} \cos(2\pi y) \sin(\pi x) \]
Numerical Examples

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Convergence of the coefficients of RBF-FD equations to usual FD equations when $h = 0.1$

<table>
<thead>
<tr>
<th>shape parameter ($c$)</th>
<th>coef. of $u_{i,j}$</th>
<th>coef. of the neighboring nodes</th>
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</thead>
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<tr>
<td>0.1</td>
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<td>20.0</td>
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<td>100.0029</td>
</tr>
<tr>
<td>30.0</td>
<td>-400.0043</td>
<td>100.0011</td>
</tr>
</tbody>
</table>
2. Poisson Equation in a unit disk

\[ \Delta u = f \text{ in } \Omega = \{(x, y)|x^2 + y^2 < 1\}, \quad u = g \text{ on } \partial\Omega \]

where \( f \) and \( g \) are computed from the exact solution,

\[ u(\bar{x}) = u(x, y) = \frac{25}{25 + (x - 0.2)^2 + 2y^2} \]
RBF-FD for scattered data
FD for uniform grid

rbf-fd ni = 5
FD 1.12e-4
rbf-fd ni = 9
rbf-cfd ni = 9 mi = 5

shape parameter (c)
Future plan

Extension to,

- non-linear convection-diffusion problems
- transient problems
References


[14] **G. Wright and B. Fornberg**, *Scattered node compact finite
difference-type formulas generated from radial basis functions., To appear in J. Comp. Phy.

Visible Research Output


THANK YOU